Mathematics 189-133B, Winter 2003
Vectors, Matrices and Geometry
Solutions to Written Assignment 1, due in class, Friday, January 24, 2003

1. Let Π_1 be the plane in R^3 defined by $\vec{n}_1 \cdot (\vec{x} - \vec{x}_0) = 0$. (So $\vec{n}_1$ is a normal vector and $\vec{x}_0$ some particular point on Π_1.) Let Π_2 be a different plane through the same point $\vec{x}_0$, with equation $\vec{n}_2 \cdot (\vec{x} - \vec{x}_0) = 0$. Show that the intersection of the two planes is the line given by the equation $\vec{x} = \vec{x}_0 + t(\vec{n}_1 \times \vec{n}_2)$.

   The assumption that the planes are different implies that the normal vectors $\vec{n}_1$ and $\vec{n}_2$ are in different directions, so that $\vec{n}_1 \times \vec{n}_2 \neq \vec{0}$.

   If $\vec{x}$ is on the intersection of the two planes, then $\vec{x} - \vec{x}_0$ is orthogonal to both $\vec{n}_1$ and $\vec{n}_2$ and hence is a multiple of $\vec{n}_1 \times \vec{n}_2$. So $\vec{x} - \vec{x}_0 = t(\vec{n}_1 \times \vec{n}_2)$ and $\vec{x}$ is on the line described.

   Conversely, if $\vec{x}$ is on the line, then $\vec{x} - \vec{x}_0 = t(\vec{n}_1 \times \vec{n}_2)$ for some scalar t. So $(\vec{x} - \vec{x}_0) \cdot \vec{n}_1 = t(\vec{n}_1 \times \vec{n}_2) \cdot \vec{n}_1 = t(0) = 0$ and $\vec{x}$ is on Π_1. Similarly, $\vec{x}$ is on Π_2, too.

2. Let A, B, C and D be any points in the plane that determine a quadrilateral $ABCD$. (We assume that these are listed in an order so that the diagonals are $\overline{AC}$ and $\overline{BD}$, and they cross.) Suppose that E, F, G and H are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ respectively. Using vector methods, show that

   (a) The quadrilateral $EFGH$ is a parallelogram.

   (b) Assuming that $ABCD$ is itself a parallelogram, show that $EFGH$ is a rectangle if and only if $ABCD$ is a rhombus.

   (c) Assuming that $ABCD$ is itself a parallelogram, show that $EFGH$ is a rhombus if and only if $ABCD$ is a rectangle.

1. $\vec{EF} = \vec{EB} + \vec{BC} = \frac{1}{2}\vec{AB} + \frac{1}{2}\vec{BC} = \frac{1}{2}\vec{AC}$, and $\vec{HG} = \vec{HD} + \vec{DG} = \frac{1}{2}\vec{AD} + \frac{1}{2}\vec{DC} = \frac{1}{2}\vec{AC}$. Thus the sides $\overline{EF}$ and $\overline{HG}$ are of the same length and in the same direction, showing that $EFGH$ is a parallelogram. We also note that $\vec{EH} = \vec{FG} = \frac{1}{2}\vec{BD}$.

2. If $ABCD$ is a parallelogram, then $\vec{AB} = \vec{DC} = \vec{u}$ and $\vec{BC} = \vec{AD} = \vec{v}$ for some vectors $\vec{u}$ and $\vec{v}$. So $\vec{EF} = \frac{1}{2}(\vec{u} + \vec{v})$ and $\vec{EH} = \frac{1}{2}(\vec{v} - \vec{u})$. (We use "iff." to abbreviate "if and only if".)

   $EFGH$ is a rectangle iff.

   $\vec{EF} \cdot \vec{EH} = 0$ iff.

   $\frac{1}{2}(\vec{u} + \vec{v}) \cdot \frac{1}{2}(\vec{v} - \vec{u}) = 0$ iff.

$\frac{1}{4}(\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} - \vec{u} \cdot \vec{u} - \vec{v} \cdot \vec{u}) = 0$ iff.

$\vec{u} \cdot \vec{u} = \vec{v} \cdot \vec{v}$ iff.

$||\vec{u}||^2 = ||\vec{v}||^2$ iff.

$||\vec{u}|| = ||\vec{v}||$ (since $||\vec{u}||$ and $||\vec{v}||$ are positive scalars) iff.

$ABCD$ is a rhombus.

3. $EFGH$ is a rhombus iff.

   $||\vec{EF}|| = ||\vec{EH}||$ iff.

   $\frac{1}{2}||\vec{u} + \vec{v}|| = \frac{1}{2}||\vec{v} - \vec{u}||$ iff.

   $||\vec{u}+\vec{v}||^2 = ||\vec{v}-\vec{u}||^2$ (since $||\vec{u}+\vec{v}||$ and $||\vec{v}-\vec{u}||$ are positive numbers) iff.

   $(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = (\vec{v} - \vec{u}) \cdot (\vec{v} - \vec{u})$ iff.

   $\vec{u} \cdot \vec{u} + 2(\vec{u} \cdot \vec{v}) + \vec{v} \cdot \vec{v} = \vec{u} \cdot \vec{u} - 2(\vec{u} \cdot \vec{v}) + \vec{v} \cdot \vec{v}$ iff.

   $4(\vec{u} \cdot \vec{v}) = 0$ iff.

   $\vec{u}$ and $\vec{v}$ are perpendicular iff.

   $ABCD$ is a rectangle.

(Note that in this problem, both directions of the "if and only if" statements can be done at once; this happens sometimes, and does not happen other times.)