MATH 133

Short Solutions to Final Exam

[1] (1) The area of the triangle is $\sqrt{\mathbf{5}}$.
(2) The equation is $x + 2z - 3 = 0$.
(3) The projection of $\vec{PR}$ onto $\vec{PQ}$ is $(-8/3, 4/3, 4/3)$.
(4) The distance between R and ℓ is $\sqrt{30}/3$.

[2] (a) The vectors are linearly independent for all k except $k = 5$ and $k = -10$.
(b) The vectors span $\mathbf{R}^3$ for all k except $k = 5$ and $k = -10$.

[3]

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

[4] (1) The standard matrix of T_1 is

$$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}.$$

(2) The standard matrix of T_2 is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(3) The standard matrix of T_3 is

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

[5] (a)

$$A = \begin{bmatrix} -11 & 10 \\ -15 & 14 \end{bmatrix}.$$

(b)

$$A^{100} = \begin{bmatrix} -2 \cdot 4^{100} + 3 & 2 \cdot 4^{100} - 2 \\ -3 \cdot 4^{100} + 3 & 3 \cdot 4^{100} - 2 \end{bmatrix}$$

[6] $P^{-1}AP = D$ where

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

[7] (1) $\det(R^2 - 2R) = -1$.
(2) $\operatorname{rank}(A) = 4$.
(3) $\det(A^{-1}) = -1/6$.
(4) $\det(\operatorname{adj}(-A^T)) = 16$.
(5) The eigenvalues of A are -6 and 1.
(6) $\det(A^2 - 3A) = 4$.

[8]

$$Q = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$

[9] Let

$$Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and let

$$\mathbf{v}_1 = \begin{bmatrix} a \\ c \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} b \\ d \end{bmatrix},$$

be the columns of Q. The columns form an orthonormal set if and only if $\mathbf{v}_1 \cdot \mathbf{v}_1 = 1$, $\mathbf{v}_2 \cdot \mathbf{v}_2 = 1$, $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

Now,

$$Q^T Q = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 \\ \mathbf{v}_1 \cdot \mathbf{v}_2 & \mathbf{v}_2 \cdot \mathbf{v}_2. \end{bmatrix}$$

Hence,

$$Q^T Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

if and only if $\mathbf{v}_1 \cdot \mathbf{v}_1 = 1$, $\mathbf{v}_2 \cdot \mathbf{v}_2 = 1$, $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. It follows that $Q^T Q = I$ if and only if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal set.

[10] This problem can be solved in two different ways.

The first solution is based on determinants. By the fundamental theorem of invertible matrices, if the columns of A are linearly dependent, then $\det(A) = 0$. Hence, $\det(AB) = \det(A)\det(B) = 0$, and again by the fundamental theorem of invertible matrices, the rows of AB are linearly dependent.

The second solution is based on solution to problem 3 on written assignment 2. Note first that by the fundamental theorem of invertible matrices, **if the columns of A are linearly dependent, then the rows of A are also linearly dependent**. After this observation, you can follow line by line solution to problem 3 on written assignment 2.

[11] (1) $n = 1$ and $m = 2$.
(2) $E_1 = \text{span}\{[1, 0, 2]\}$.
(3) $E_{-3} = \text{span}\{[1, -2, 2]\}$.
(4) The geometric multiplicity of the eigenvalue 1 is 1 and the geometric multiplicity of the eigenvalue -3 is 1.
(5) From (1) and (4) we see that the geometric multiplicity of the eigenvalue -3 is different from its algebraic multiplicity. Hence, the matrix A is not diagonalizable.

[12] (1) A basis for W is $\{[1, 0, 1], [0, 1, 1]\}$.
(2) A basis for $W^\perp$ is $\{[-1, -1, 1]\}$.
(3) An orthogonal basis for W is $\{[1, 0, 1], [-1/2, 1, 1/2]\}$.
(4) $\text{proj}_W(\mathbf{v}) = [0, 1, 1]$.