MATH 133: Vectors, Matrices and Geometry

Solution to Written Assignment 6

Solution.

(a) $\operatorname{tr}(aA+bB)=\sum_{i=1}^n(aa_{ii}+bb_{ii})=a\sum_{i=1}^n a_{ii}+b\sum_{i=1}^n b_{ii}=a\operatorname{tr}(A)+b\operatorname{tr}(B)$.

$$\operatorname{tr}(AB)=\sum_{i,j=1}^n a_{ij}b_{ji}=\sum_{i,j=1}^n a_{ji}b_{ij}=\sum_{i,j=1}^n b_{ij}a_{ji}=\operatorname{tr}(BA).$$

Since $\operatorname{tr}(AB-BA)=\operatorname{tr}(AB)-\operatorname{tr}(BA)=0$ and $\operatorname{tr}(I)=n$, we see that $AB-BA=I$ is impossible for $n\geq 1$.

(b) If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$A^2-\operatorname{tr}(A)A+\det(A)=\begin{bmatrix} a^2+bc-(a+d)a+ad-bc & ab+bd-(a+d)b \\ ac+dc-(a+d)c & bc+d^2-(a+d)d+ad-bc \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(c) $A[1,1,\ldots,1]^T=[c_1,c_2,\ldots,c_n]^T$ where $c_i=a_{i1}+a_{i2}+\ldots+a_{in}$ is the sum of the entries in the i-th row of A. Hence $A[1,1,\ldots,1]^T=c[1,1,\ldots,1]^T \iff c=c_1=c_2=\cdots=c_n$.

(d) The determinant is unchanged if we add 10^i times the i-th column to the first for $i=1,2,3,4$. The resulting determinant then has the given numbers as entries of the last column.