MATH 133: Vectors, Matrices and Geometry

Solution to Written Assignment 6

Solution.

(a) $\mathrm{tr}(aA+bB) = \sum_{i=1}^{n}(aa_{ii}+bb_{ii}) = a\sum_{i=1}^{n} a_{ii} + b\sum_{i=1}^{n} b_{ii} = a\mathrm{tr}(A) + b\mathrm{tr}(B)$.

$$\mathrm{tr}(AB) = \sum_{i,j=1}^{n} a_{ij}b_{ji} = \sum_{i,j=1}^{n} a_{ji}b_{ij} = \sum_{i,j=1}^{n} b_{ij}a_{ji} = \mathrm{tr}(BA).$$

Since $\mathrm{tr}(AB-BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$ and $\mathrm{tr}(I) = n$, we see that $AB - BA = I$ is impossible for $n \geq 1$.

(b) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$A^2 - \mathrm{tr}(A)A + \det(A) = \begin{bmatrix} a^2+bc-(a+d)a+ad-bc & ab+bd-(a+d)b \\ ac+dc-(a+d)c & bc+d^2-(a+d)d+ad-bc \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(c) $A[1,1,\ldots,1]^T = [c_1, c_2, \ldots, c_n]^T$ where $c_i = a_{i1} + a_{i2} + \ldots + a_{in}$ is the sum of the entries in the i-th row of A. Hence $A[1,1,\ldots,1]^T = c[1,1,\ldots,1]^T \iff c = c_1 = c_2 = \cdots = c_n$.

(d) The determinant is unchanged if we add 10^i times the i-th column to the first for $i = 1,2,3,4$. The resulting determinant then has the given numbers as entries of the last column.