MATH 133: Vectors, Matrices and Geometry

Solution to Written Assignment 5

Solution. This problem uses the fact that if $A_1, A_2, \dots, A_p$ are the columns of A and $X = [x_1, x_2, \dots, x_p]^T$ then

$$AX = x_1 A_1 + x_2 A_2 + \cdots + x_p A_p.$$

(a) The j-th column of AB is AB_j, where B_j is the j-th column of B. Thus the columns of AB are linear combinations of columns of A which implies that the column space of AB (the span of the columns of AB) is contained in the column space of A (the span of the columns of A).

(b) Since $\mathrm{col}(AB) \subseteq \mathrm{col}(A)$ by (a), we have $\mathrm{rank}(AB) = \dim \mathrm{col}(AB) \leq \dim \mathrm{col}(A) = \mathrm{rank}(A)$. We also have

$$\mathrm{rank}(AB) = \mathrm{rank}((AB)^T) = \mathrm{rank}(B^T A^T) \leq \mathrm{rank}(B^T) \leq \mathrm{rank}(B).$$

(c) It suffices to show $BC = I$ implies that $\mathrm{col}(AB) = \mathrm{col}(A)$. If $Z \in \mathrm{col}(A)$ then $Z = AY$ for some $Y \in \mathbb{R}^p$. But then $Z = AY = ABCY = AB(X)$ where $X = CY$ which shows that $Z \in \mathrm{col}(AB)$. Thus $\mathrm{col}(A) \subseteq \mathrm{col}(AB)$ and we are done by (a).

(d) If $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ then $AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B^2 = 0$ and $\mathrm{rank}(AB) = \mathrm{rank}(A) = 1$.

Bonus Problem.

(a) If I_1, I_2 are the columns of the identity 2×2 matrix I then columns C_1, C_2 of C satisfy $BC_i = I_i$. Solving these equations, we find a solution

$$C = \begin{bmatrix} 3 & -2 \\ -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(b) We have $\mathrm{rank}(B) = p \iff \mathrm{col}(B) = \mathbb{R}^p \iff BX = Y$ is solvable for X for any Y in $\mathbb{R}^p$ by the statement we made at the beginning. If $BX = Y$ is solvable for any Y in $\mathbb{R}^p$ then, taking for Y the columns of the identity $p \times p$ matrix, we find as solutions the columns of a matrix C with $BC = I$. Conversely, if $BC = I$ and $Y \in \mathbb{R}^p$ is given then $X = CY$ satisfies $BX = Y$.