MATH 133: Vectors, Matrices and Geometry

Solution to Written Assignment 4

Solution.

(a) Since the given planes $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ intersect in a line L, their normal vectors (a_1, b_1, c_1), (a_2, b_2, c_2) are non-proportional and so are linearly independent. The linear combination of the given equations

$$A(a_1x + b_1y + c_1z) + B(a_2x + b_2y + c_2z) = Ad_1 + Bd_2,$$

obtained by adding A times the first equation to B times the second, is satisfied by any common solution of the two given equations and so by any point of line L. This equation can be written

$$(a_1A + a_2B)x + (b_1A + b_2B)y + (c_1A + c_2B)z = d_1A + d_2A.$$

This is the equation of a plane if the vector $\overrightarrow{n} = (a_1A + a_2B, b_1A + b_2B, c_1A + c_2B)$ is not the zero vector. But

$$\overrightarrow{n} = A(a_1, b_1, c_1) + B(a_2, b_2, c_2)$$

which is not zero if A, B are not both zero. Therefore to solve the problem we want to show that, given a point $P_0(x_0, y_0, z_0)$ not on L, we can find A, B not both zero such P_0 lies on the plane $\overrightarrow{n} \cdot (x, y, z) = d_1A + d_2B$. The latter holds if and only if

$$A(a_1x_0 + b_1y_0 + c_1z_0) + B(a_2x_0 + b_2y_0 + c_2z_0) = Ad_1 + Bd_2,$$

or, equivalently, $A(a_1x_0 + b_1y_0 + c_1z_0 - d_1) + B(a_2x_0 + b_2y_0 + c_2z_0 - d_2) = 0$. Since (x_0, y_0, z_0) is not on L, not both $s = a_1x_0 + b_1y_0 + c_1z_0 - d_1$ and $t = a_2x_0 + b_2y_0 + c_2z_0 - d_2$ are zero and so $A = t, B = -s$ is a non-zero solution.

To find the equation of the plane passing through point $(1, 2, 3)$ and containing the line of intersection of the planes $2x + 3y - 4z = 1$, $3x - 2y + 4z = 1$ we take

$$A(2x + 3y - 4z) + B(3x - 2y + 4z) = A + B,$$

where $-5A + 10B = 0$. For example, take $A = 2, B = 1$ so that the required plane is $7x + 4y - 4z = 3$.

(b) If three planes $a_1x + b_1y + c_z = d_1, a_2x + b_2y + c_2z = d_2, a_3x + b_3y + c_3z = d_3$ have no point in common, the system of equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

has no solution which implies that the coefficient matrix of the above system is row equivalent to a matrix with a zero row. Since the row space of a matrix is unchanged by elementary row operations, this means the the normal vectors of the given planes are in a subspace which can be spanned by two vectors, i.e., the normal vectors lie in a plane.