MATH 133: Vectors, Matrices and Geometry

Solution to Written Assignment 3

Solution.

(a) The statement is false. For example, take $z = -w$. The given vectors are then $u-w, v-w, 0$ which are linearly dependent since $0(u-w) + 0(v-w) + 1 \cdot 0 = 0$ is a nontrivial dependence relation.

(b) The statement is false since the vectors $u + 2v - 3w, 2u + v - w, 3u - v + 2w$ are linearly independent. Indeed,

$$a(u + 2v - 3w) + b(2u + v - w) + c(3u - v + 2w) = 0 \implies (a + 2b + 3c)u + (2a + b - c)v + (-3a - b + 2c)w = 0$$

which implies $a+2b+3c = 2a+b-c = -3a-b+2c = 0$ since u, v, w are linearly independent. Solving this homogeneous system for a, b, c, we get

$$\begin{aligned} a + 2b + 3c &= 0 \\ 2a + b - c &= 0 \\ -3a - b + 2c &= 0 \end{aligned} \quad \begin{matrix} E_2 - 2E_1 \\ E_3 + 3E_1 \end{matrix} \quad \begin{aligned} a + 2b + 3c &= 0 \\ -3b - 7c &= 0 \\ 5b + 11c &= 0 \end{aligned} \quad E_3 + (5/3)E_2 \quad \begin{aligned} a + 2b + 3c &= 0 \\ -3b - 7c &= 0 \\ (-2/3)c &= 0 \end{aligned}$$

which implies $a = b = c = 0$ and hence the linear independence of the given vectors.

(c) The statement is false since the given vectors are linearly dependent. Indeed,

$$a(u+v+w)+b(u+2v+3w)+c(2u+3v+w)+d(3v+u+2w) = 0 \iff (a+b+2c+d)u+(a+2b+3c+3d)v+(a+3b+c+2d)w = 0.$$

Now the homogeneous system of equations

$$\begin{aligned} a + b + 2c + d &= 0 \\ a + 2b + 3c + 3d &= 0 \\ a + 3b + c + 2d &= 0 \end{aligned}$$

has a non-zero solution since there are more unknowns than equations and this implies the existence of a non-trivial dependence relation among the given vectors. To get an explicit dependence relation, we solve the system to get the equivalent system $a = 2d, b = -d, c = -d$ which has d as a free variable. Setting $d = 1$, we get the solution $a = 2, b = -1, c = -1, d = 1$. This yields the non trivial dependence relation

$$2(u + v + w) - (u + 2v + 3w) - (2u + 3v + w) + (3v + u + 2w) = 0.$$