MATH 133: Vectors, Matrices and Geometry

Solution to Written Assignment 1

Problem. Using the properties of inner products, prove that the diagonals of a parallelogram are orthogonal if and only if the sides of the parallelogram are equal in length.

Solution. If $\mathbf{u}, \mathbf{v}$ are the sides of the parallelogram, then $\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}$ are its diagonals. We have

$$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v} = ||\mathbf{u}||^2 - ||\mathbf{v}||^2 \quad \text{since} \quad \mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$$

so that $||\mathbf{u}|| = ||\mathbf{v}|| \iff ||\mathbf{u}||^2 = ||\mathbf{v}||^2 \iff (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = 0 \iff \mathbf{u}+\mathbf{v} \perp \mathbf{v}-\mathbf{v}$.