Solutions for Test 2

1. The characteristic polynomial of A is $\det(A - \lambda I) = -\lambda^3 + 13\lambda^2 - 36\lambda = -\lambda(\lambda - 4)(\lambda - 9)$.

2. The eigenvalues of A are $\lambda = 0, 4, 9$, the roots of the characteristic polynomial.

3. The eigenspaces of A are

$$E_0 = \text{null}(A) = \text{span}([1, -2, 1]^T), \quad E_4 = \text{null}(A - 4I) = \text{span}([1, 0, -1]^T), \quad E_9 = \text{null}(A - 9I) = \text{span}([1, 1, 1]^T.$$

4. If $P = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ then $P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$.

5. If $B = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & \pm 2 & 0 \\ 0 & 0 & \pm 3 \end{bmatrix} P^{-1}$ then $B^2 = A$.

6. If X is an eigenvector of A, say $AX = dX$, and $AB = BA$ then $ABX = BAX = BdX = dX$ which states that BX is in the eigenspace $E_d = \text{span}(X)$ and hence that $BX = \lambda X$. Thus every eigenvector of A is an eigenvector of B. Hence

$$PBP^{-1} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

so that $B^2 = A \iff a^2 = 0, b^2 = 4, c^2 = 9$ which implies that B is one of the four matrices found in 4.