Solutions for Test 1

1. (a) Since a direction vector for L is $\overrightarrow{AB} = (1, -2, 5)$ and $(1, -2, 3)$ is a point on L, the vector form of the parametric equations for L is $(x, y, z) = (1, -2, 3) + t(1, -2, 5)$ or, equivalently $x = 1 + t, y = -2 - 2t, z = 3 + 5t$.

   (b) The orthogonal projection of the vector $\overrightarrow{AP} = (-1, -3, 4)$ onto $\overrightarrow{AB} = (1, -2, 5)$ is the vector

$$\overrightarrow{AQ} = \frac{\overrightarrow{AP} \cdot \overrightarrow{AB}}{\overrightarrow{AB} \cdot \overrightarrow{AB}} \overrightarrow{AB} = \frac{25}{30}(1, -2, 5) = (5/6, -5/3, 25/6).$$

   The distance from P to L is

$$||\overrightarrow{PQ}|| = ||\overrightarrow{AQ} - \overrightarrow{AP}|| = ||(11/6, 4/3, 1/6)|| = \sqrt{186}/6.$$

   (c) The point Q on L closest to P has coordinate vector

$$\overrightarrow{OA} + \overrightarrow{AQ} = (2, 1, -1) + (5/6, -5/3, 25/6) = (17/6, -2/3, 19/6).$$

   (d) Since the plane has normal vector $(1, -2, 5)$ and passes through $(1, -2, 3)$, it has the equation $x - 2y + 5z = 20$.

2. (a) Solving the two equations by Gaussian elimination, you get in matrix form

$$\begin{bmatrix} 1 & -2 & 3 & 2 \\ 2 & 1 & -3 & 3 \end{bmatrix} R_2 - 2R_1 \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 5 & -9 & -1 \end{bmatrix} \frac{1}{5}R_2 \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & -9/5 & -1/5 \end{bmatrix} R_1 + 2R_2 \begin{bmatrix} 1 & 0 & -3/5 & 8/5 \\ 0 & 1 & -9/5 & -1/5 \end{bmatrix}$$

   so that our equations in reduced echelon form are $x = 8/5 + 3z/5, y = -1/5 + 9z/5$. Setting $z = t$ as our parameter, we get $x = 8/5 + 3t/5, y = -1/5 + 9t/5, z = t$ as parametric equations for L or, in vector form, $(x, y, z) = (8/5, -1/5, 0) + t(3/5, 9/5, 1)$.

   (b) Taking $t = -1$ in the parametric equations for L we find that $A(1, -2, -1)$ is a point of L. A direction vector for L is $\overrightarrow{d} = (3, 9, 5)$ so that a normal for the required plane is

$$\overrightarrow{d} \times \overrightarrow{AP} = (3, 9, 5) \times (0, 1, 2) = (13, -6, 3).$$

   Its equation is therefore $13x - 6y + 3z = 22$.

   (c) We want to write $(13, -6, 3, 22)$ in the form $a(1, -2, 3, 2) + b(2, 1, -3, 3)$ which is equivalent to solving the system of equations

$$\begin{aligned} a + 2b &= 13 \\ -2a + b &= -6 \\ 3a - 3b &= 3 \\ 2a + 3b &= 22 \end{aligned}$$

   for a, b. Using Gaussian elimination, we get

$$\begin{bmatrix} 1 & 2 & 13 \\ -2 & 1 & -6 \\ 3 & -3 & 3 \\ 2 & 3 & 22 \end{bmatrix} \begin{matrix} R_2 - 2R_2 \\ R_3 - 3R_1 \\ R_4 - 2R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 13 \\ 0 & 5 & 20 \\ 0 & -9 & -36 \\ 0 & -1 & -4 \end{bmatrix} \begin{matrix} \frac{1}{5}R_2 \\ \frac{-1}{9}R_3 \\ -R_4 \end{matrix} \begin{bmatrix} 1 & 2 & 13 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ R_3 - R_2 \\ R_4 - R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   from which we get $a = 5, b = 4$.

3. Solving the system by Gaussian elimination in matrix form, we get

$$\begin{bmatrix} 1 & c & 2 & c & 1 \\ 2 & 2c & 5 & 1 & c+1 \\ c & c^2 & 3c & -2 & c+3 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_2 - cR_1 \end{matrix} \begin{bmatrix} 1 & c & 2 & c & 1 \\ 0 & 0 & 1 & 1-2c & c-1 \\ 0 & 0 & c & -2-c^2 & c+3 \end{bmatrix} R_3 - cR_2 \begin{bmatrix} 1 & c & 2 & c & 1 \\ 0 & 0 & 1 & 1-2c & c-1 \\ 0 & 0 & 0 & c^2-c-2 & -(c^2-c-2) \end{bmatrix}$$

Since $c^2 - c - 2 = (c-2)(c+1)$, we have three cases: $c = -1$, $c = 2$ and $c \neq -1, 2$. If $c = -1$, our system is

$$\begin{bmatrix} 1 & -1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_1 - 2R_2 \begin{bmatrix} 1 & -1 & 0 & -7 & 5 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

from which $x_1 = x_2 + 7x_4 + 5, x_3 = -3x_4 - 2$. Setting $x_2 = s, x_4 = t$ as parameters we get the parametric form for the the solutions

$$x_1 = s + 7t + 5, x_2 = s, x_3 = -3t - 2, x_4 = t.$$

When $c = 2$ we get

$$\begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_1 - 2R_2 \begin{bmatrix} 1 & 2 & 0 & 8 & -1 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

from which we get the parametric representation $x_1 = -2s - 8t - 1, x_2 = s, x_3 = 3t + 1, x_4 = t$. When $c \neq -1, 2$ we can divide the last equation by $c^2 - c - 2$ to get

$$\begin{bmatrix} 1 & c & 2 & c & 1 \\ 0 & 0 & 1 & 1-2c & c-1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{matrix} R_1 - cR_3 \\ R_2 - (1-2c)R_3 \end{matrix} \begin{bmatrix} 1 & c & 2 & 0 & c+1 \\ 0 & 0 & 1 & 0 & -c \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} R_1 - 2R_2 \begin{bmatrix} 1 & c & 0 & 0 & 3c+1 \\ 0 & 0 & 1 & 0 & -c \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

from which we get the parametric representation

$$x_1 = 3c + 1 - cs, x_2 = s, x_3 = -c, x_4 = t.$$

We thus see that the system is consistent for all values of c.