Linear dependence and independence

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10:12 AM

If $u_1, u_2, \ldots, u_m \in \mathbb{R}^n$ then

$$\text{span}(u_1, \ldots, u_n) = \{ s_1 u_1 + \cdots + s_m u_m \mid s_1, \ldots, s_m \in \mathbb{R} \}$$

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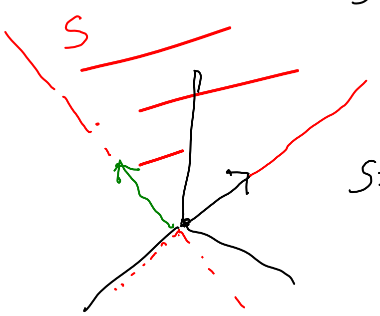
$s_1 u_1 + \cdots + s_m u_m$ linear combination of the vector $u_1, \ldots, u_m$

Examples:



$$\text{span}([-1,2,3]) = \{ s[-1,2,3] \mid s \in \mathbb{R} \}$$
$$= \{ [-s, 2s, 3s] \mid s \in \mathbb{R} \}$$

$$S = \text{span}([-1,2,3], [1,2,4])$$
$$= \{ s[-1,2,3] + t[1,2,4] \mid s,t \in \mathbb{R} \}$$

$$= \{ [-s+t, 2s+2t, 3s+4t] \mid s,t \in \mathbb{R} \}$$

$[x_1, x_2, x_3] \in \text{span}([-1,2,3], [1,2,4])$

$\Longleftrightarrow$
$$x_1 = -s+t$$
$$x_2 = 2s+2t \qquad s,t \in \mathbb{R}$$
$$x_3 = 3s+4t$$

these are the parametric equations of a plane in $\mathbb{R}^3$ going through the origin

Problem. Determine whether or not

$$\mathbb{R}^3 = \text{span}([-1,2,3], [1,2,4])$$

Solution. $[x_1, x_2, x_3] \in \text{span}([-1,2,3], [1,2,4])$

$\iff [x_1, x_2, x_3] = s[-1, 2, 3] + t[1, 2, 4]$ for some $s, t \in \mathbb{R}$

$$\iff \begin{aligned} -s + t &= x_1 \\ 2s + 2t &= x_2 \\ 3s + 4t &= x_3 \end{aligned}$$

$$\iff \begin{aligned} -s + t &= x_1 & & \\ 4t &= x_2 + 2x_1 & & (E_2 + 2E_1) \\ 7t &= x_3 + 3x_1 & & (E_3 + 3E_1) \end{aligned}$$

$$\iff \begin{aligned} -s + t &= x_1 & & \\ t &= \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 & & (\tfrac{1}{4}E_2) \\ t &= \tfrac{3}{7}x_1 + \tfrac{1}{7}x_3 & & (\tfrac{1}{7}E_3) \end{aligned}$$

$$\iff \begin{aligned} -s + t &= x_1 & & \\ t &= \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 & & \\ 0 &= -\tfrac{1}{14}x_1 - \tfrac{1}{4}x_2 + \tfrac{1}{7}x_3 & & (E_3 - E_2) \end{aligned}$$

This system has a solution

$$\iff -\tfrac{1}{14}x_1 - \tfrac{1}{4}x_2 + \tfrac{1}{7}x_3 = 0$$

normal equation of a plane.

$\therefore$ Answer to problem is no; the given vectors don't span since, for example,

$$[1, 0, 0] \notin \operatorname{span}([-1, 2, 3], [1, 2, 4])$$

Using the normal equation, we can find another spanning set as follows:

$$x_1 = -\frac{14}{4}x_2 + \frac{14}{7}x_3 = -\frac{7}{2}x_2 + 2x_3$$

~~If~~ Let $x_2 = t_1$, $x_3 = t_2$, then

$x_1 = -\frac{7}{2}t_1 + 2t_2$ so solution set

$$= \{ [-\tfrac{7}{2}t_1 + 2t_2, t_1, t_2] \mid t_1, t_2 \in \mathbb{R} \}$$

$$\shortparallel$$

$$t_1 \underbrace{[-\tfrac{7}{2}, 1, 0]}_{u} + t_2 \underbrace{[2, 0, 1]}_{v}$$

$$\text{span}([-1,2,3],[1,2,4]) = \text{span}([-\tfrac{7}{2}, 1, 0], [2,0,1])$$

The set $S = \text{span}(u_1, u_2, \ldots, u_m)$ has the following properties

(1) $\vec{0} \in S$. $\quad \vec{0} = \underset{\text{Zero scalar}}{0} \cdot \vec{u}_1 + 0 \cdot \vec{u}_2 + \ldots + 0 \cdot \vec{u}_m$

(2) $\left.\begin{matrix} u, v \in S \\ a, b \in \mathbb{R} \end{matrix}\right\} \Rightarrow au + bv \in S$

Indeed, $u = s_1 u_1 + s_2 u_2 + \ldots + s_m u_m \in S \qquad s_i, t_i \in \mathbb{R}$

$v = t_1 u_1 + t_2 u_2 + \ldots + t_m u_m \in S$

$\Rightarrow au + bv = (as_1 + bt_1)u_1 + \cdots + (as_m + bt_m)u_m \in S$

$\text{span}(u_1, \ldots, u_m)$ is called the subspace spanned or generated by the vectors $u_1, \ldots, u_m$.

$\{u_1, u_2, \ldots, u_m\}$ is called the spanning or generating set.

Another important example of a subspace is the solution set of a system of homogeneous equations

Consider the general system of m linear homogeneous equations in n unknowns $x_1, \ldots, x_n$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\vdots \qquad \qquad & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

and let S be its solution set, namely, the set of all $[x_1, x_2, \ldots, x_n] \in \mathbb{R}^n$ which satisfy these equations. Then $S \subseteq \mathbb{R}^n$ We want to show S is a subspace of $\mathbb{R}^n$.

(1) We have $0 = [0, 0, \ldots, 0] \in S$ since the system is homogeneous.

(2) If $[x_1, x_2, \ldots, x_n], [y_1, y_2, \ldots, y_n] \in S$ and $a, b \in \mathbb{R}$ then $a[x_1, x_2, \ldots, x_n] + b[y_1, y_2, \ldots, y_n]$

$$= [ax_1 + by_1, ax_2 + by_2, \ldots, ax_n + by_n] \quad \text{and}$$

$$
\begin{aligned}
& a_{i1}(ax_1 + by_1) + a_{i2}(ax_2 + by_2) + \cdots + a_{in}(ax_n + by_n) \\
&= a(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n) + b(a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{in}y_n) \\
&= a \cdot 0 + b \cdot 0 = 0
\end{aligned}
$$

which shows that $[ax_1 + by_1, \ldots, ax_n + by_n]$ satisfies each equation of our system and hence that $a[x_1, \ldots, x_n] + b[y_1, \ldots, y_n] \in S$. Therefore S is a subspace of $\mathbb{R}^n$

Example. $x_1 + x_2 + x_3 + x_4 = 0$

$$x_3 + x_4 = 0$$

$E_1 - E_2$: $\quad x_1 + x_2 = 0 \quad$ or $\quad x_1 = -x_2$

$$x_3 + x_4 = 0 \qquad x_3 = -x_4$$

$$\text{Solution set} = \left\{ [-s, s, -t, t] \mid s, t \in \mathbb{R} \right\}$$

$$[-s, s, -t, t] = s[-1, 1, 0, 0] + t[0, 0, -1, 1]$$

$\therefore$ solution set $= \mathrm{span}([-1, 1, 0, 0], [0, 0, -1, 1])$

Vectors $u_1, u_2, \ldots, u_m$ are linearly dependent

$\iff$ there are scalars $t_1, \ldots, t_m$ not all 0

such that $t_1 u_1 + t_2 u_2 + \cdots + t_m u_m = 0$

$\iff$ some $u_i \in \mathrm{span}(u_1, \ldots, \hat{u}_i, \ldots, u_m)$

The vectors $u_1, u_2, \ldots, u_m$ are linearly independent

$\iff$ they are not linearly dependent

$\iff$ $t_1 u_1 + \cdots + t_m u_m = 0 \implies t_1 = t_2 = \cdots = t_m = 0$

$\iff$ $u_i \notin \mathrm{span}(u_1, \ldots, \hat{u}_i, \ldots, u_m)$ for all i