Linear dependence and independence

Wednesday, October 01, 2003
11:14 AM

Before we begin our ~~topic for today~~ let's do a problem of the type that was given to you on the second written assignment.

Problem. Determine the nature of the solution set of the system

$$tx_1 + x_2 = 1$$
$$x_1 + tx_2 = 1$$

for the various values of the parameter t.

Solution. The augmented matrix of this system is

$$\begin{bmatrix} t & 1 & 1 \\ 1 & t & 1 \end{bmatrix}$$

Row reducing this matrix, we get

$$\begin{bmatrix} t & 1 & 1 \\ 1 & t & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & t & 1 \\ t & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - tR_1} \begin{bmatrix} 1 & t & 1 \\ 0 & 1-t^2 & 1-t \end{bmatrix}$$

Case 1: $t \neq \pm 1$. In this case $1-t^2 \neq 0$ so we can multiply the last row of the above matrix on the right by $1/1-t^2$ to get

$$\begin{bmatrix} 1 & t & 1 \\ 0 & 1 & 1/1+t \end{bmatrix}$$

This is the coefficient matrix of the system

$$x_1 + t x_2 = 1$$
$$x_2 = 1/1+t$$

Back substituting, we get

$$x_1 = 1 - t x_2 = 1 - \frac{t}{1+t} = \frac{1}{1+t}$$

so the system has the unique solution $x_1 = x_2 = \frac{1}{1+t}$.

Case II: $t = 1$. In this case, our system has the same solution set as the system

$$x_1 + x_2 = 1$$
$$0 = 0$$

Setting $x_2 = s$, we get the solutions

$$[x_1, x_2] = [1-s, s] = [1, 0] + s[-1, 1]$$

which is the vector equation of a line in $\mathbb{R}^2$

Case 3: $t = -1$. In this case the system has the same solution set as the system

$$x_1 - x_2 = 1$$
$$0 = 2$$

which is an inconsistent system. Therefore there are no solutions in this case

Consider the general system of m linear equations in the n unknowns $x_1, x_2, \dots, x_n$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad & \cdot \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

where the scalars a_{ij} $(1 \le i \le m,\ 1 \le j \le n)$, $b_1, b_2, \dots, b_m$ are given. The augmented matrix of this system is

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_1 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{bmatrix}
$$

m rows

$n+1$ columns

and the coefficient matrix is

$$
A = [a_{ij}] = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
$$

m rows

n columns

We can write the above system as a single equation in $\mathbb{R}^m$:

$$
x_1 \underset{C_1}{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}} + x_2 \underset{C_2}{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}} + \cdots x_n \underset{C_n}{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}} = \underset{b}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}
$$

If we define the column space of the matrix $A = [a_{ij}]$ to be the set of all possible linear combinations of the columns $C_1, C_2, \ldots, C_n$ of A

i.e. $\text{Columnspace}(A) = \{ x_1 C_1 + x_2 C_2 + \cdots + x_n C_n \mid x_1, x_2, \ldots, x_n \in \mathbb{R} \}$.

then the problem of solving our system is equivalent to the problem of determining whether a given column vector b is in the column space of A which is a subset of $\mathbb{R}^m$

Similarly, one can define the row space of A as the set of all possible linear combinations of the rows of A. Note that

$$\text{row space}(A) \subseteq \mathbb{R}^n.$$

The row and column spaces are examples of subspaces of a vector space. A subset S of $\mathbb{R}^n$ is said to be a subspace of $\mathbb{R}^n$ is

(1) the zero vector of $\mathbb{R}^n$ is in S

(2) $u, v \in S,\ a, b \in \mathbb{R} \Rightarrow au + bv \in S$

An important example of a subspace of $\mathbb{R}^n$ is the set all possible linear combinations of a given sequence of vectors $u_1, u_2, \ldots, u_m$ in $\mathbb{R}^n$. This set is denoted by $\text{span}(u_1, u_2, \ldots, u_n)$ and is called the span of $u_1, u_2, \ldots, u_n$.

The set $\text{span}(u_1, u_2, \dots, u_m) = \{x_1 u_1 + \dots + x_m u_m \mid x_i \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^n$ since

(1) $0 \cdot u_1 + 0 \cdot u_2 + \dots + 0 \cdot u_n =$ the zero vector of $\mathbb{R}^n$

(2) $u = x_1 u_1 + \dots + x_m u_m$, $v = y_1 u_1 + \dots + y_m u_m$, $a, b \in \mathbb{R}$

$\Rightarrow au + bv = (ax_1 + by_1) u_1 + \dots + (a x_m + b y_m) u_m$.

The subspace $\text{span}(u_1, u_2, \dots, u_m)$ is called the subspace spanned or generated by $u_1, u_2, \dots, u_m$.

In order to be able to define what is meant by the dimension of a subspace, we need to define the concepts of linear independence and linear dependence.

Definition. The vectors $u_1, u_2, \dots, u_m$ are said to be *linearly independent* if none of the vectors u_i is a linear combination of the other vectors, i.e.

(A) $\quad u_i \notin \text{span}(u_1, u_2, \dots, \hat{u_i}, \dots, u_m)$

where $\hat{u_i}$ means that the vector u_i is omitted.

This is equivalent to

(B) $\quad x_1 u_1 + x_2 u_2 + \dots + x_m u_m = 0 \Rightarrow x_1 = x_2 = \dots = x_m = 0$

the zero vector

Let's prove the equivalence of (A) and (B)

(A) $\Rightarrow$ (B). Here we are assuming (A) and we are trying to prove (B). If (B) were false, then there would exist scalars $x_1, x_2, \ldots, x_m$ not all zero such that $x_1 u_1 + x_2 u_2 + \cdots + x_m u_m = 0$. Suppose $x_1 \neq 0$. Then multiplying both sides by x_1^{-1} we get

$$u_1 + x_1^{-1} x_2 u_2 + \cdots + x_1^{-1} x_m u_m = 0$$

or, solving for u_1, $\quad u_1 = (-x_1^{-1} x_2) u_2 + \cdots + (-x_1^{-1} x_m) u_m$
contradicting $u_1 \notin \text{span}(u_2, u_3, \ldots, u_m)$. A similar argument works if $x_i \neq 0$ for some $i > 1$. So (B) must be true.

(B) $\Rightarrow$ (A). Here we assume (B) and try to prove (A). If (A) were false, say $u_i \in \text{span}(u_1, \ldots, \hat{u}_i, \ldots, u_m)$, then

$$u_i = a_1 u_1 + \cdots + a_{i-1} u_{i-1} + a_{i+1} u_{i+1} + \cdots + a_m u_m$$

for some scalars a_i. But then

$$a_1 u_1 + \cdots + a_{i-1} u_{i-1} + (-1) u_i + a_{i+1} u_{i+1} + \cdots + a_m u_m = 0$$

where the coefficient of u_i is $-1 \neq 0$. This contradicts (B) so (A) must be true.

Definition. The vectors $u_1, u_2, \ldots, u_m$ are said to be *linearly dependent* if they are not linearly independent.

Now let's do some numerical problems illustrating these concepts.

Problem 1. Determine the linear dependence or independence of the vectors

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad u_4 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad u_5 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}$$

Solution. We want to find those scalars x_1, x_2, x_3, x_4, x_5 such that $x_1 u_1 + x_2 u_2 + x_3 u_3 + x_4 u_4 + x_5 u_5 = 0$. This equation is equivalent to the system of 4 equations in 5 unknowns x_1, x_2, x_3, x_4, x_5

$$\begin{aligned}
x_1 + x_2 + 2x_3 + x_4 + 2x_5 &= 0 \\
2x_1 + x_2 + x_3 + 2x_4 + 3x_5 &= 0 \\
3x_1 + x_2 + 2x_3 + 2x_4 + x_5 &= 0 \\
4x_1 - x_2 + x_3 + 2x_4 + 2x_5 &= 0
\end{aligned}$$

We want to determine whether or not this homogeneous system as a non-zero solution or not. This can be decided by Gaussian elimination. In this case, we don't have to go through this process as we have shown that a homogeneous system of linear equations with more unknowns than equations has a non-zero solution. The given vectors u_1, u_2, u_3, u_4, u_5 are therefore linearly dependent.

Problem 2. Determine whether or not the vectors $u_1 = [1, 2\ 3]$, $u_2 = [2, 1, 4]$, $u_3 = [3, 3, 7]$ are linearly dependent or independent

Solution. Here we see by inspection that $u_3 = u_1 + u_2$. So u_1, u_2, u_3 are linearly dependent. If we hadn't noticed this we would proceed as follows

$x_1 u_1 + x_2 u_2 + x_3 u_3 = 0$ is equivalent to the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 0 \\ 3x_1 + 4x_2 + 7x_3 &= 0 \end{aligned}$$

Row reducing the coefficient matrix to echelon form we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 4 & 7 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & -2 & -2 \end{bmatrix} R_3 - R_2 \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} (-\tfrac{1}{3})R_3 \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This gives the equivalent system $\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$

which has x_3 as a free variable. So non-zero solutions exist and the vectors are linearly dependent. To get an explicit dependence relation we take $x_3 = 1$ to get the solution $x_1 = -1$, $x_2 = -1$, $x_3 = 1$. Hence $-u_1 - u_2 + u_3 = 0$ or $u_3 = u_1 + u_2$.