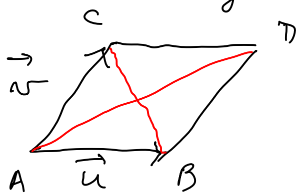
Problem. Show that a parallelogram is a rectangle $\iff$ the diagonals have the same length.



$\vec{u} = \overrightarrow{AB}$, $\vec{v} = \overrightarrow{AC}$

diagonals $\overrightarrow{AD} = \vec{u} + \vec{v}$, $\overrightarrow{BC} = \vec{v} - \vec{u}$

$\|\overrightarrow{AD}\| = \|\overrightarrow{BC}\| \iff \|\overrightarrow{AD}\|^2 = \|\overrightarrow{BC}\|^2$

$\iff (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$

$\cancel{\vec{u}\cdot\vec{u}} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \cancel{\vec{v}\cdot\vec{v}} = \cancel{\vec{u}\cdot\vec{u}} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \cancel{\vec{v}\cdot\vec{v}}$

$\iff 2\vec{u}\cdot\vec{v} = -2\vec{u}\cdot\vec{v}$

$\iff 4\vec{u}\cdot\vec{v} = 0 \iff \vec{u}\cdot\vec{v} = 0$

A matrix is in echelon form if

(1) The zero rows come after the non-zero rows

(2) if the first non-zero entry in row i occurs in column j_i, then $j_1 < j_2 < \dots$.
The columns $j_1, j_2, \dots$ are called pivot columns and the leading entries called pivots

Ex $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ in echelon form

$j_1 = 1$, $j_2 = 2$ $\qquad j_1 = 1$, $j_2 = 3$

A matrix is in reduced echelon form if in addition to conditions (1) and (2) we have

(3) the pivot columns have exactly one non-zero entry

(4) the leading entries are all $\underline{1}$

Example: $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ reduced echelon form

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ red. ech. form

Possible 2×3 matrices in reduced echelon form:

$$\begin{bmatrix}1&0&*\\0&1&*\end{bmatrix},\ \begin{bmatrix}1&*&0\\0&0&1\end{bmatrix},\ \begin{bmatrix}0&1&0\\0&0&1\end{bmatrix},\ \begin{bmatrix}1&*&*\\0&0&0\end{bmatrix},\ \begin{bmatrix}0&1&*\\0&0&0\end{bmatrix},\ \begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}$$

$j_1=1, j_2=2$; $j_1=1, j_2=3$; $j_1=2, j_2=3$; $j_1=1$; $j_1=2$; $j_1=1$

$$\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$$

Theorem. Any matrix can be brought to echelon or reduced form by a finite number of elementary operations.

Example Solve

$$\begin{aligned}x_1+x_2+x_3&=1\\2x_1+x_2+2x_3&=2\\3x_1+2x_2+3x_3&=3\end{aligned}$$

$$\begin{bmatrix}1&1&1&1\\2&1&2&2\\3&2&3&3\end{bmatrix}\xrightarrow[R_3-3R_1]{R_2-2R_1}\begin{bmatrix}1&1&1&1\\0&-1&0&0\\0&-1&0&0\end{bmatrix}\xrightarrow{R_3-R_2}\begin{bmatrix}1&1&1&1\\0&-1&0&0\\0&0&0&0\end{bmatrix}$$

augmented matrix of the system

$$\xrightarrow{R_1+R_2}\begin{bmatrix}1&0&1&1\\0&-1&0&0\\0&0&0&0\end{bmatrix}\xrightarrow{-R_2}\begin{bmatrix}1&0&1&1\\0&1&0&0\\0&0&0&0\end{bmatrix}$$

∴ system becomes $x_1+x_3=1$, $x_2=0$; $x_1=1-x_3$, $x_2=0$

∴ Solution set $=\{(1-t,0,t)\mid t\in\mathbb{R}\}$

$[x_1,x_2,x_3]=[1,0,0]+t[-1,0,1]$

Derive general sol. of

$$\begin{aligned}ax+by&=e\\cx+dy&=f\end{aligned}$$

Case I: $a\neq 0$

$$\begin{bmatrix}a&b&e\\c&d&f\end{bmatrix}\xrightarrow{\frac{1}{a}R_1}\begin{bmatrix}1&b/a&e/a\\c&d&f\end{bmatrix}\xrightarrow{R_2-cR_1}\begin{bmatrix}1&\frac{b}{a}&e/a\\0&d-\frac{bc}{a}&f-ce/a\end{bmatrix}$$

$$\begin{bmatrix}1&b/a&e/a\\0&D/a&D_2/a\end{bmatrix}\qquad D=ad-bc,\quad D_2=af-ce$$

If $D\neq 0$, then $y=D_2/D$, $x=e/a-(b/a)(D_2/D)=D_1/D$ where $D_1=ed-bf$

$D=\begin{vmatrix}a&b\\c&d\end{vmatrix}=ad-bc$, $D_1=\begin{vmatrix}e&b\\f&d\end{vmatrix}=ed-bf$, $D_2=\begin{vmatrix}a&e\\c&f\end{vmatrix}=af-ec$

So $x=\frac{D_1}{D}$, $y=\frac{D_2}{D}$ unique solution if $D\neq 0$.

So $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ unique solution if $D \neq 0$.
This is called Cramer's Rule.

If $D = 0$, and $D_2 = 0$ have ∞ # of sol
If $D = 0$, $D_2 \neq 0$ have no sol
since last equation is $0 = 1$ after
mult by a const. (inconsistent equation)

Case II $a = 0$, $c \neq 0$ interchange eqns to get to
Case I.

Case III $a = 0$, $c = 0$ Poss. red. echelon forms

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$$

no Sol. infinity of solutions

Theorem. The reduced echelon form of a matrix
is unique

The number of non-zero rows in the reduced
echelon form of a matrix A is called
the rank of A.
If A is the coefficient matrix of a system
of equations in n unknowns and $r =$
rank of A then system can be solved
for r of the variables $x_{j_1}, x_{j_2}, \ldots, x_{j_r}$
where $j_1, j_2, \ldots, j_r$ are the pivot cols
in term of the remaining $n - r$ variables

If the system of equations is homogeneous
i.e. constants on the right hand side of the
equations are all zero

eg $x_1 + x_2 + 2x_3 = 0$, $2x_1 + x_3 = 0$, $x_1 + x_2 = 0$

Such a system is always consistent since

Such a system is always consistent since $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is a solution, the zero solution.

Theorem: A homogeneous system of linear equations with more unknowns than equations always has a non-zero solution

If $n = \#$ of equations, $r =$ rank of coeff matrix we have $r < n$. Can solve system for $x_{j_1}, x_{j_2}, \ldots, x_{j_r}$ in terms of the $n-r$ remaining variables.

$\Rightarrow$ there exist non-zero solutions

e.g. set the remaining variables anything you like (non-zero in particular) + solve for $x_{j_1}, \ldots, x_{j_r}$ to get a non-zero solution

Example:

$$
\begin{aligned}
x_1 &= 2x_3 + 3x_5 \\
x_2 &= x_3 - x_5 \\
x_4 &= x_5
\end{aligned}
$$

Set $x_3 = 1, x_5 = 0$ get sol $[2, 1, 1, 0, 0]$

Solution set $= \{ [2s+3t, s-t, s, t, t] \mid s, t \in \mathbb{R} \}$

General solution:

$[x_1, x_2, x_3, x_4, x_5] = s[2, 1, 1, 0, 0] + t[3, -1, 0, 1, 1]$