Gauss-Jordan Elimination

Example:
$$\begin{aligned} 2x + 3y + z &= 1 \\ x + y + 2z &= 2 \end{aligned} \qquad (*)$$

Find all solutions of these equations, i.e find all (x, y, z) satisfying $(*)$

$E_1 \leftrightarrow E_2$
$$\begin{aligned} x + y + 2z &= 2 \\ 2x + 3y + z &= 1 \end{aligned}$$

$E_2 \to E_2 - 2E_1$
$$\begin{aligned} x + y + 2z &= 2 \\ y - 3z &= -3 \end{aligned}$$
echelon form

$E_1 \to E_1 - E_2$
$$\begin{aligned} x + 5z &= 5 \\ y - 3z &= -3 \end{aligned}$$
reduced-echelon form

Solutions: $x = 5 - 5t$, $y = -3 + 3t$, $z = t$ (t arb)

Solution set $= \{ (5-5t, -3+3t, t) \mid t \text{ arb. scalar} \}$

Write this in matrix form:

$$\begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 3 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & -3 & -3 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 5 & 5 \\ 0 & 1 & -3 & -3 \end{bmatrix}$$

echelon form (Gauss reduced)

reduced echelon form (Gauss-Jordan reduced)

Describe this process (Gauss-Jordan Elimination) in general

Elementary operations on $\left\{ \begin{matrix} \text{equations} \\ \text{matrices} \end{matrix} \right\}$: 3 types

I: Interchange two $\left\{ \begin{matrix} \text{equations} \\ \text{rows} \end{matrix} \right\}$

II: Add a multiple of one $\left\{ \begin{matrix} \text{equation} \\ \text{row} \end{matrix} \right\}$ to another $\left\{ \begin{matrix} \text{eqn} \\ \text{row} \end{matrix} \right\}$

III: Multiply an $\left\{ \begin{matrix} \text{equation} \\ \text{row} \end{matrix} \right\}$ by a non-zero scalar

These operations on equations don't change the solution set of the system since these elementary operations are reversible solution set of the system since ... ... of operations are reversible

$\underline{I}$ $E_i \leftrightarrow E_j$ inverse: $E_i \leftrightarrow E_j$

$\underline{II}$ $E_i \to E_i + aE_j$ ($j \neq i$) inverse: $E_i \to E_i - aE_j$

$\underline{III}$ $E_i \to cE_i$ $(c \neq 0)$ inverse. $E_i \to \frac{1}{c} E_i$

Description of elementary operations on matrices

$\underline{III}$ $R_i \to cR_i$ $R_i = [a_1, \dots, a_n]$, $cR_i = [ca_1, \dots, ca_n]$

$\underline{II}$ $R_i \to R_i + cR_j$ $R_j = [b_1, \dots, b_n]$ $R_i + cR_j = [a_1 + cb_1, \dots, a_n + cb_n]$

if $c = 1$ then $R_i + R_j = [a_1 + b_1, \dots, a_n + b_n]$

$\mathbb{R}^n = \{[a_1, \dots, a_n] \mid a_1, \dots, a_n \in \mathbb{R}\}$

called the space of n-dimensional (numerical) vectors

Why do we need $\mathbb{R}^n$?

Problem: Find all curves $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ that pass through $(1,1), (2,3), (3,-1)$

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 &= 1 \\ a_0 + 2a_1 + 4a_2 + 8a_3 &= 3 \\ a_0 + 3a_1 + 9a_2 + 27a_3 &= -1 \end{aligned}$$

linear system of 3 equations in 4 unknowns a_0, a_1, a_2, a_3

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 &= 1 \\ E_2 - E_1 \qquad a_1 + 3a_2 + 7a_3 &= 2 \\ E_3 - E_1 \qquad 2a_1 + 8a_2 + 26a_3 &= -2 \end{aligned}$$

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 &= 1 \\ E_3 - 2E_2 \qquad a_1 + 3a_2 + 7a_3 &= 2 \\ 2a_2 + 12a_3 &= -6 \end{aligned}$$

a_0, a_1, a_2 leading variables

a_3 remaining variable

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 & 2 \\ 0 & 0 & 2 & 12 & -6 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 & 2 \\ 0 & 0 & 1 & 6 & -3 \end{bmatrix}$$

$$\xrightarrow[R_2 - 3R_3]{R_1 - R_3} \begin{bmatrix} 1 & 1 & 0 & -5 & 4 \\ 0 & 1 & 0 & -11 & 11 \\ 0 & 0 & 1 & 6 & -3 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & 6 & -7 \\ 0 & 1 & 0 & -11 & 11 \\ 0 & 0 & 1 & 6 & -3 \end{bmatrix}$$

reduced echelon form

reduced echelon form

$$a_0 + 6a_3 = -7$$
$$a_1 - 11a_3 = 11$$
$$a_2 + 6a_3 = -3$$

system of equations with same solution set as original system.

Solutions: $a_0 = -7 - 6t$
$a_1 = 11 + 11t$
$a_2 = -3 - 6t$
$a_3 = t$
(t arbitrary scalar)

or $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 11 \\ -3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ 11 \\ -6 \\ 1 \end{bmatrix}$

a line in $\mathbb{R}^4$

A matrix is in <u>reduced echelon form</u> if

(1) the non-zero rows preceed the zero rows

(2) the first non-zero entry in any row is $\underline{1}$.

(3) In the column where the first non-zero entry appears, there is exactly one non-zero entry
(These columns are called pivot columns)

(4) If the leading entry in row i occurs in column j_i, then $j_1 < j_2 < j_3 < \cdots$
(Columns $j_1, j_2, j_3, \ldots$ are the pivot columns)