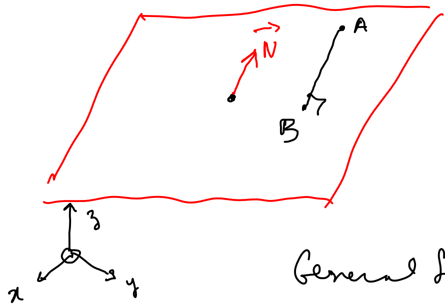
Begin with a problem: Find the distance from the point $A(1,1,1)$ to the plane $x+2y+3z=1$.



$\vec{N} = [1,2,3]$

$$D = \|\vec{AB}\|$$

$$= \left|\frac{1+2+3-1}{\sqrt{1^2+2^2+3^2}}\right| = \frac{5}{\sqrt{14}}$$

General formula for D

$A(x_1, y_1, z_1)$ $\quad ax+by+cz=d$

$$D = \left|\frac{ax_1+by_1+cz_1-d}{\sqrt{a^2+b^2+c^2}}\right|$$

Second method for finding D:

Line joining AB passes through $A(1,1,1)$ and has direction $\vec{N} = [1,2,3]$. $\therefore$ Parametric equations of the line are

$$x = 1+t, \quad y = 1+2t, \quad z = 1+3t$$

Substitute in $x+2y+3z=1$ to get

$$1+t + 2(1+2t) + 3(1+3t) = 1$$

$$6 + 14t = 1$$

$$t = -\frac{5}{14}$$

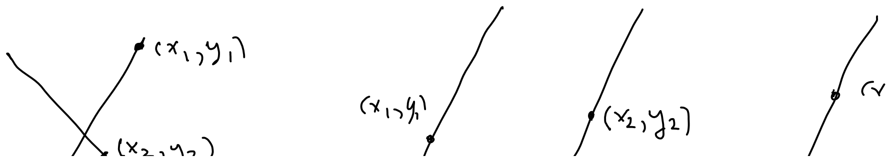
$\therefore$ Coords of B are

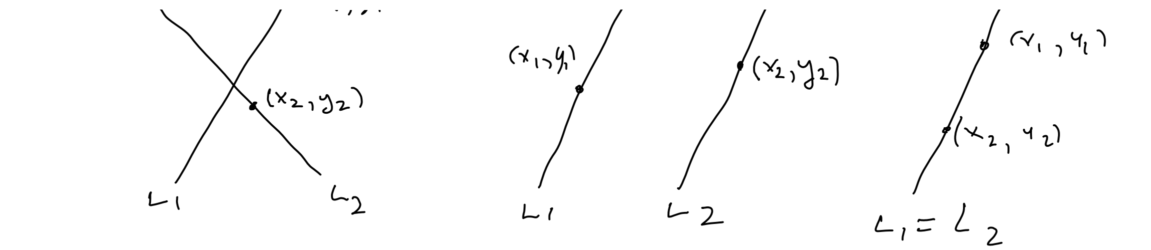
$$\left(1-\frac{5}{14}, 1-\frac{10}{14}, 1-\frac{15}{14}\right) = \left(\frac{9}{14}, \frac{4}{14}, \frac{-1}{14}\right)$$

$$D = \|\vec{AB}\| = \sqrt{\left(1-\frac{9}{14}\right)^2 + \left(1-\frac{4}{14}\right)^2 + \left(1+\frac{1}{14}\right)^2}$$

$$= \frac{5}{\sqrt{14}}$$

Interection of 2 lines L_1, L_2 in a plane: 3 possibilities





Par eqns of L_1 & L_2:

L_1: $[x,y] = [x_1 + t\alpha, y_1 + t\beta]$

L_2: $[x,y] = [x_2 + t\gamma, y_2 + t\delta]$

Points of intersection of L_1 and L_2 satisfy

$$x_1 + t\alpha = x_2 + s\gamma$$
$$y_1 + t\beta = y_2 + s\delta$$

i.e. $\left.\begin{array}{l} t\alpha - s\gamma = x_2 - x_1 \\ t\beta - s\delta = y_2 - y_1 \end{array}\right\}$ linear equations in s, t

Setting $a = \alpha$, $b = \beta$, $c = -\gamma$, $d = -\delta$, $e = x_2 - x_1$, $f = y_2 - y_1$ we get

E_1: $at + cs = e$

E_2: $bt + ds = f$

General form of a system of linear equations in 2 unknowns.. s, t

Since $[a, b, 0] \times [c, d, 0] = [0, 0, ad - bc]$ we see that $[a, b]$ and $[c, d]$ are not parallel if and only if $ad - bc \neq 0$. In this case, the system has a unique solution since the lines L_1, L_2 are not // and hence meet in a unique pt. Let us find s, t in this case.

$aE_2 - bE_1$: $(da - bc)s = af - be$

$\therefore s = \dfrac{af - be}{ad - bc}$. similarly, $t = \dfrac{ed - cf}{ad - bc}$

Define $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Then $t = \dfrac{\begin{vmatrix} e & c \\ f & d \end{vmatrix}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}}$, $s = \dfrac{\begin{vmatrix} a & e \\ b & f \end{vmatrix}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}}$ Cramer's Rule

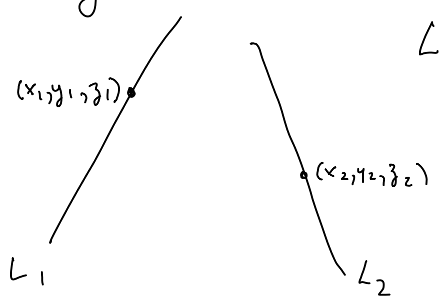
If $L_1 // L_2$ then $[c, d] = r[a, b]$ for some $r > 0$ and our system becomes $\begin{array}{l} at + ras = e \\ bt + rbs = f \end{array}$. If $b = 0$, then $f = 0$; otherwise, the system has no solution (this is the case $L_1 \neq L_2$).

If $b \neq 0$ then subtracting a/b times the 2nd equation from the first, we get $0 = e - af/b$. If $e \neq af/b$, this otherwise the system

If $b \neq 0$ then subtracting a/b times the 2nd equation from the first, we get $0 = e - af/b$. If $e \neq af/b$, this equation is inconsistent and so our system has no solution. This is the case $L_1 \parallel L_2$, $L_1 \neq L_2$. If $e = af/b$, then our system reduces to a single equation. This is the case $L_1 = L_2$ and we have an infinity of solutions.

In the plane, a line L with parametric equations $x = x_0 + \alpha t$, $y = y_0 + \beta t$ can be given by a single equation $ax + by = c$ by solving one of the equations for t and substituting in the second equation. If (x_0, y_0) is on L then this equation can be written $a(x - x_0) + b(y - y_0) = 0$ which shows that $[a, b] \perp L$. Thus $ax + by = c$ called the normal equation of L.

<u>Case of two lines L_1, L_2 in 3-space:</u>



$$L_1: \quad \begin{aligned} x &= x_1 + t\alpha_1 \\ y &= y_1 + t\beta_1 \\ z &= z_1 + t\gamma_1 \end{aligned}$$

$$L_2: \quad \begin{aligned} x &= x_2 + t\alpha_2 \\ y &= y_2 + t\beta_2 \\ z &= z_2 + t\gamma_2 \end{aligned}$$

These two lines meet $\Leftrightarrow$ we can find scalars s, t such that

$$\begin{aligned} x_1 + \alpha_1 t &= x_2 + \alpha_2 s \\ y_1 + \beta_1 t &= y_2 + \beta_2 s \\ z_1 + \gamma_1 t &= z_2 + \gamma_2 s \end{aligned}$$

The lines L_1 and L_2 are parallel

$$\Leftrightarrow [\alpha_1, \beta_1, \gamma_1] \times [\alpha_2, \beta_2, \gamma_2] = 0$$

i.e. $$\left[\begin{vmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{vmatrix}, -\begin{vmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \end{vmatrix}, \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \right] = 0$$

If L_1 and L_2 are not parallel then one of these determinants is $\neq 0$ and so at least one pair of equations in (**) can be solved for s, t. If these values satisfy the remaining equation then there is a unique solution (the lines meet in a unique pt) If not, the lines don't meet and the lines [illegible]

solution ("the lines meet in a unique pt")
If not, the lines don't meet and the lines are said to be skew lines (nonparallel lines which don't meet).
If L_1 and L_2 are parallel then $L_1 = L_2 \iff$ (x_1, y_1, z_1) lies on L_2 or (x_2, y_2, z_2) lies on L_1.

Example: To find the points of intevection of the lines

L_1: $x = 1+t,\ y = 1+2t,\ z = 1+3t$

L_2: $x = 1-t,\ y = 4+t,\ z = 6+2t$

we want to find scalars s, t such that

$$\begin{aligned} 1+t &= 1-s \\ 1+2t &= 4+s \\ 1+3t &= 6+2s \end{aligned} \qquad \text{i.e.} \qquad \begin{aligned} t+s &= 0 \\ 2t-s &= 3 \\ 3t-2s &= 5 \end{aligned}$$

Since $\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3 \neq 0$, we can solve the first two equations to get $t = \dfrac{\begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}} = \dfrac{-3}{-3} = 1$, $s = \dfrac{\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}} = \dfrac{3}{-3} = -1$

Substituting in the third equation, we get $5 = 5$ which means that $t = 1, s = -1$ is the unique solution. This yields $(2, 3, 4)$ as the unique point of intersection of L_1 and L_2.