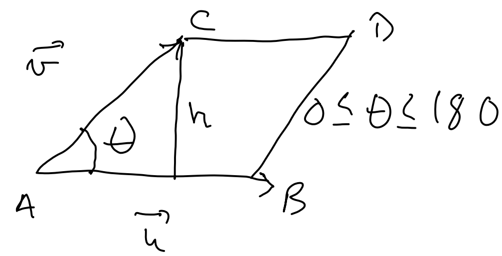
Dot Products:

$\vec{u} = \overrightarrow{AB}$, $\vec{v} = \overrightarrow{AC}$



$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

$\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$

$\theta = \arccos\left(\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}\right)$ if θ given in radians (π radians $= 180°$)

$= \dfrac{180}{\pi} \arccos\left(\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}\right)$ if θ given in degrees

$\vec{u} = [x_1, y_1, z_1]$, $\vec{v} = [x_2, y_2, z_2]$

$\vec{u} \cdot \vec{v} = x_1 x_2 + y_1 y_2 + z_1 z_2$

$\vec{u} \cdot \vec{u} = x_1^2 + y_1^2 + z_1^2 = \|\vec{u}\|^2$

Cross Product of $\vec{u}$ and $\vec{v}$

$\vec{u} \times \vec{v}$ is the unique vector such that

(1) $\vec{u} \times \vec{v} \perp \vec{u}$ and $\vec{v}$

(2) $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta$

$= $ area of $\|$gm ABCD

$= 2$ area $\triangle ABC$

(3) $\vec{u}, \vec{v}, \vec{u} \times \vec{v}$ right handed triple

if you rotate $\vec{u}$ into $\vec{v}$ using right hand then thumb points in the direction of $\vec{u} \times \vec{v}$

$\vec{u} = [x_1, y_1, z_1]$

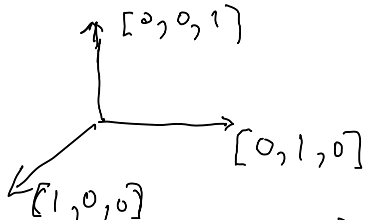
$$\vec{v} = [x_2, y_2, z_2]$$

$$\vec{u} \times \vec{v} = [y_1 z_2 - y_2 z_1,\ x_2 z_1 - x_1 z_2,\ x_1 y_2 - x_2 y_1]$$

Exercise: Prove (1) $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$
and $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$

(2) $\|\vec{u \times v}\|^2 + |\vec{u} \cdot \vec{v}|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2$

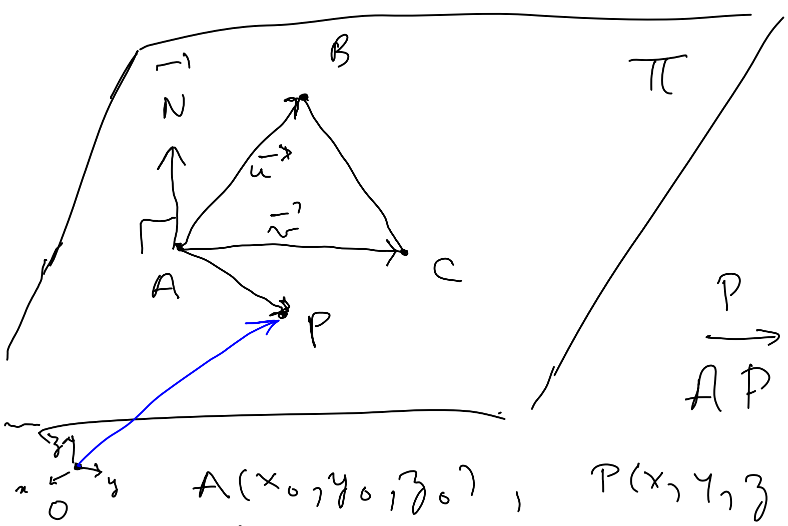
$$[1,0,0] \times [0,1,0] = [0,0,1]$$



$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

$$\vec{u} \times \vec{u} = [0,0,0]$$

$$(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$$



Normal Equation of a Plane π

$\vec{N} = \vec{u} \times \vec{v} \perp \pi$

P lies on π $\Leftrightarrow$

$$\vec{AP} \cdot \vec{N} = 0$$

$A(x_0, y_0, z_0)$, $P(x, y, z)$, $\vec{N} = (a, b, c)$

$$\vec{AP} \cdot \vec{N} = [x - x_0, y - y_0, z - z_0] \cdot [a, b, c]$$
$$= a(x - x_0) + b(y - y_0) + c(z - z_0)$$
$$= ax + by + cz - d$$

$\therefore$ where $d = ax_0 + by_0 + cz_0$

$\therefore$ normal equation of π is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$\therefore$ $ax + by + cz - d = 0$

or $\quad ax + by + cz - d = 0$

or $\quad ax + by + cz = d$

Example: $A(1,1,1)$, $B(2,2,2)$, $C = (3,4,3)$

Find the normal equation of the plane passing through A, B, C

$\vec{u} = \vec{AB} = [1,1,1]$, $\vec{v} = \vec{AC} = [2,3,2]$

$$\vec{u} = [1,1,1]$$
$$\vec{v} = [2,3,2]$$

$\vec{u} \times \vec{v} = [-1, 0, 1] = \vec{N}$ normal to plane

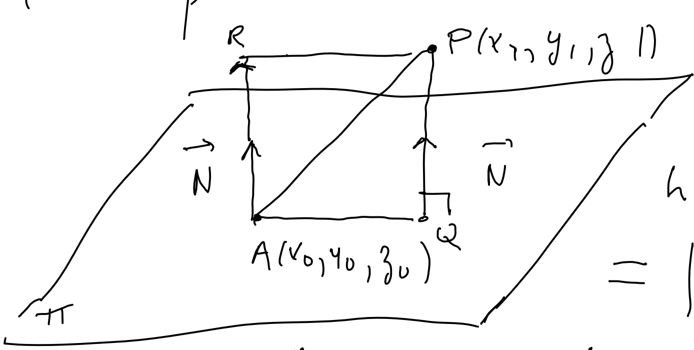
$\therefore$ normal equation of plane is

$$(-1)x + (0)y + (1)z = (-1)(1) + (0)(1) + (1)(1)$$
$$= 0$$

$ax + by + cz = d$ unique up to mult. by a non-zero scalar

origin is on the plane $\Leftrightarrow d = 0$

Formula for the distance of a point to a plane π



$\vec{PQ}$ = projection of $\vec{AP}$ along $\vec{N}$

$h = \|\vec{PQ}\| = \|\vec{AR}\| \qquad \vec{N} = [a,b,c]$

$$= \left\| \frac{\vec{AP} \cdot \vec{N}}{\vec{N} \cdot \vec{N}} \vec{N} \right\|$$

$$= \left\| \frac{(x_1 - x_0)a + (y_1 - y_0)b + (z_1 - z_0)c}{a^2 + b^2 + c^2} [a,b,c] \right\|$$

$$= \left\| \frac{(x_1-x_0)a+(y_1-y_0)b+(z_1-z_0)c}{a^2+b^2+c^2}[a,b,c] \right\|$$

$$= \left\| \frac{x_1a+y_1b+z_1c-d}{\sqrt{a^2+b^2+c^2}} \underbrace{\frac{[a,b,c]}{\sqrt{a^2+b^2+c^2}}}_{} \right\|$$

$$= \frac{|ax_1+by_1+cz_1-d|}{\sqrt{a^2+b^2+c^2}}$$

unit vector or vector of length 1

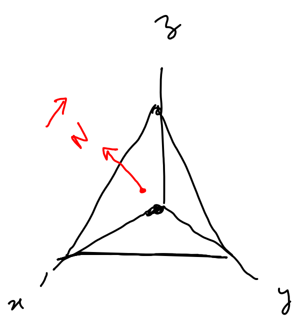
$ax_1+by_1+cz_1-d>0 \iff P$ lies on same side of π as $\vec{N}$

Example: Let $A=(1,0,0)$, $B=(0,1,0)$, $C=(0,0,1)$

Then $\vec{u}=\vec{AB}=[-1,1,0]$, $\vec{v}=\vec{AC}=[-1,0,1]$

$\vec{N}=\vec{u}\times\vec{v}=[1,1,1]$ and so the normal equation of the plane passing through A, B, C is

$$x+y+z=1$$



The distance from $(0,0,0)$ to this plane is the absolute value of

$$\frac{-1}{\sqrt{3}} \quad \text{i.e.} \quad \frac{1}{\sqrt{3}}$$

The minus sign tells us that the origin is below the plane

$\vec{N}$ points to the region above the plane