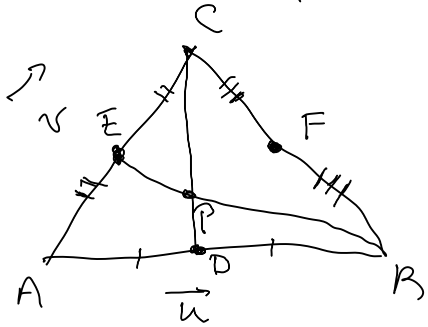
Problem. Prove that the medians of a Δ meet in a point which trisects the medians.



$\vec{u} = \vec{AB}$, $\vec{v} = \vec{AC}$

$\vec{AD} = \frac{1}{2}\vec{u}$, $\vec{DC} + \vec{AD} = \vec{AC} = \vec{v}$

$\vec{DC} = \vec{v} - \frac{1}{2}\vec{u}$

$\vec{AP} = \frac{1}{2}\vec{u} + s(\vec{v} - \frac{1}{2}\vec{u})$

$= \frac{1}{2}\vec{v} + t(\vec{u} - \frac{1}{2}\vec{v})$

$\frac{1}{2}\vec{u} + s(\vec{v} - \frac{1}{2}\vec{u}) = \frac{1}{2}\vec{u} + s\vec{v} - \frac{s}{2}\vec{u} = \frac{1-s}{2}\vec{u} + s\vec{v}$

$\frac{1}{2}\vec{v} + t(\vec{u} - \frac{1}{2}\vec{v}) = t\vec{u} + \frac{1-t}{2}\vec{v}$

$\therefore \quad \frac{1-s}{2}\vec{u} + s\vec{v} = t\vec{u} + \frac{1-t}{2}\vec{v}$

$\therefore \quad \frac{1-s}{2} = t, \quad s = \frac{1-t}{2}$

$1 - s = 2t, \quad 2s = 1 - t$

$$\boxed{\begin{aligned} s + 2t &= 1 \\ 2s + t &= 1 \end{aligned}}$$

$E_2 - 2E_1$: $s + 2t = 1$

$-3t = -1$

$\Rightarrow t = \frac{1}{3}, \quad s = \frac{1}{3}$

$\therefore \quad \vec{AP} = \frac{1}{3}\vec{u} + \frac{1}{3}\vec{v}$

$\vec{AF} = \vec{u} + \frac{1}{2}(\vec{v} - \vec{u}) = \frac{1}{2}\vec{u} + \frac{1}{2}\vec{v}$

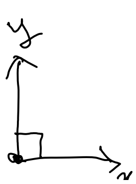
$\frac{2}{3}\vec{AF} = \frac{1}{3}\vec{u} + \frac{1}{3}\vec{v} = \vec{AP}$

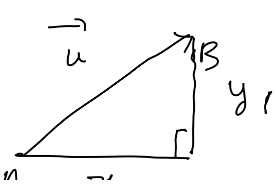
P = centroid of ΔABC

Inner Products of vectors

Inner of geometric vectors relates lengths and angles.

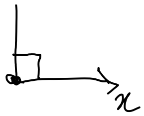
$\|\vec{u}\|$ = length or norm of $\vec{u}$

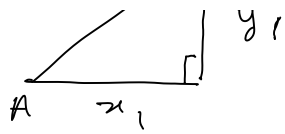




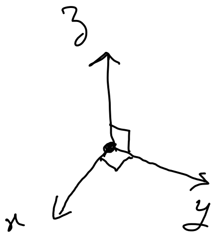
$\vec{u} = [x_1, y_1]$

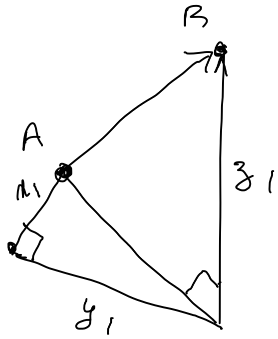
$\|\vec{u}\| = \sqrt{x_1^2 + y_1^2}$





$$\|\vec{u}\| = \sqrt{x_1^2 + y_1^2}$$





$$\vec{u} = \vec{AB} = [x_1, y_1, z_1]$$

$$h = \|\vec{AC}\|$$

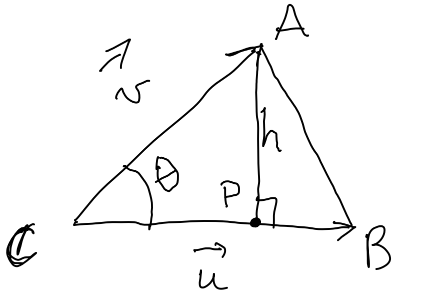
$$h^2 = y_1^2 + x_1^2$$

$$\|AB\|^2 = h^2 + z_1^2$$

$$= x_1^2 + y_1^2 + z_1^2$$

$$\|\vec{AB}\| = \|[x_1, y_1, z_1]\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

Introduce the angle between two vectors



$$\|\vec{u}\| = a, \quad \|\vec{v}\| = b$$

$$\|\vec{AB}\| = c = \|\vec{v} - \vec{u}\|$$

Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$x = \|\vec{CP}\|, \quad \frac{x}{b} = \cos\theta, \quad h = \|\vec{AP}\|$$

$$x^2 + h^2 = b^2, \quad (a-x)^2 + h^2 = c^2$$

Subtract these two equations

$$a^2 - 2ax + \cancel{x^2} + \cancel{h^2} - \cancel{x^2} - \cancel{h^2} = c^2 - b^2$$

$$\therefore c^2 = a^2 + b^2 - 2ax = a^2 + b^2 - 2ab\cos\theta$$

$\vec{u} \cdot \vec{v}$ = inner product or dot product of $\vec{u}$ and $\vec{v}$

$$\overset{def}{=} \|\vec{u}\|\|\vec{v}\|\cos\theta$$

The law of cosines can be reformulated as

$$\|\vec{u} - \vec{v}\|^2 = \|u\|^2 + \|v\|^2 - 2\vec{u} \cdot \vec{v}$$

The law of cosines can be reformulated as

$$\|\vec{u}-\vec{v}\|^2 = \|u\|^2 + \|v\|^2 - 2\,\vec{u}\cdot\vec{v}$$

$$\therefore\ \vec{u}\cdot\vec{v} = \frac{1}{2}\left\{\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2\right\}$$

$$\vec{u} = [x_1, y_1, z_1], \quad \vec{v} = [x_2, y_2, z_2]$$

$$\frac{1}{2}\left\{\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u}-v\|^2\right\} = \frac{1}{2}\left\{x_1^2 + y_1^2 + z_1^2 + x_2^2 + y_2^2 + z_2^2 - (x_1-x_2)^2 - (y_1-y_2)^2 - (z_1-z_2)^2\right\}$$

$$= x_1x_2 + y_1y_2 + z_1z_2 = \vec{u}\cdot\vec{v}$$

$$[x_1, y_1, z_1]\cdot[x_2, y_2, z_2] = x_1x_2 + y_1y_2 + z_1z_2$$

$$\vec{CP} = \frac{x}{a}\vec{u} = \frac{\|\vec{v}\|\cos\theta}{\|\vec{u}\|}\vec{u}$$

$$= \frac{\|\vec{u}\|\,\|\vec{v}\|\cos\theta}{\|\vec{u}\|^2}\vec{u}$$

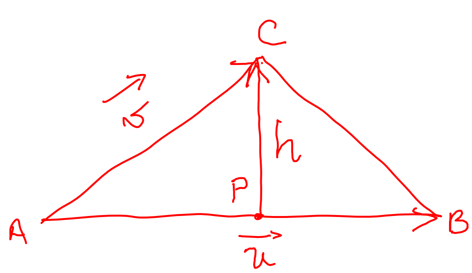
$$= \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u} \qquad \|\vec{u}\|^2 = \vec{u}\cdot\vec{u}$$

$= $ orthog proj of $\vec{v}$ along $\vec{u}$

$$= \mathrm{proj}_{\vec{u}}\vec{v}$$

$$\mathrm{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u} = \vec{AP}$$

$$h = \|\vec{PC}\| = \|\vec{v} - \mathrm{proj}_{\vec{u}}\vec{v}\|$$



$\vec{u} = \vec{AB}$

$\vec{v} = \vec{AC}$

Example. $A(1,1,1)$, $B(2,2,2)$, $C(1,2,2)$
rectangular coordinates

$\vec{u} = \vec{AB} = [1,1,1]$ $\qquad \vec{u}\cdot\vec{v} = 1\cdot 0 + 1\cdot 1 + 1\cdot 1 = 2$

$v = \vec{AC} = [0, 1, 1]$

$\vec{u}$ not perpendicular to $\vec{v}$ since $\vec{u} \cdot \vec{v} \neq 0$

$\vec{u} \cdot \vec{v} = 0 \iff \vec{u} \perp \vec{v}$

(if and only if)

$$\text{proj}_{\vec{u}} \vec{v} = \frac{2}{3}[1, 1, 1]$$

$$\vec{v} - \text{proj}_{\vec{u}} \vec{v} = [0, 1, 1] - \frac{2}{3}[1, 1, 1]$$

$$= \left[-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right]$$

$$h = \| \vec{v} - \text{proj}_{\vec{u}} \vec{v} \| = \left\| \frac{1}{3}[-2, 1, 1] \right\|$$

$$= \frac{1}{3} \| [-2, 1, 1] \|$$

$$= \frac{\sqrt{6}}{3}$$

$$\text{Area of } \Delta ABC = \frac{1}{2} \|\vec{u}\| \cdot h$$

$$= \frac{1}{2}\sqrt{3} \cdot \frac{\sqrt{6}}{3} = \frac{\sqrt{2}}{2}$$

Properties of Inner Products

1. $\vec{u} \cdot \vec{u} \geq 0$ equality $\iff \vec{u} = 0$
2. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
3. $(\vec{u}_1 + \vec{u}_2) \cdot \vec{v} = \vec{u}_1 \cdot \vec{v} + \vec{u}_2 \cdot \vec{v}$
4. $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$

Proof: $\vec{u}_1 = [x_1, y_1]$, $\vec{u}_2 = [x_2, y_2]$, $\vec{v} = [x, y]$

$$(\vec{u}_1 + \vec{u}_2) \cdot \vec{v} = [x_1 + x_2, y_1 + y_2] \cdot [x, y]$$

$$= (x_1 + x_2)x + (y_1 + y_2)y$$

$$= x_1 x + x_2 x + y_1 y + y_2 y$$

$$= (x_1 x + y_1 y) + (x_2 x + y_2 y)$$

$$= [x_1, y_1] \cdot [x, y] + [x_2, y_2] \cdot [x, y]$$

$$= [x_1, y_1] \cdot [x, y] + [x_1, y_2] \cdot [x, y]$$

$$(c[x_1, y_1]) \cdot [x, y] = [cx_1, cy_1] \cdot [x, y]$$
$$= (cx_1)x + (cy_1)y$$
$$= c(x_1 x + y_1 y) = c[x_1, y_1] \cdot [x, y]$$

this proves properties 3 and 4 for planar vectors. The case of vectors in 3-space is left to the reader