Theorem: If A is a symmetric matrix then there is an orthogonal matrix P such that

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$$P^{-1}AP$$

is a diagonal matrix.

Example $\quad A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

char polyn. of $A = \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1)$

$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ orthog to $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

eigenvector with eigenvalue 1

Indeed, $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ orthog matrix and

$$P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = P^TAP$$

Application to quadratic forms:

$$2x_1^2 + 2x_1x_2 + 2x_2^2 = [x_1, x_2]\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$[x_1, x_2]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1, x_2]\begin{bmatrix} ax_1 + bx_2 \\ bx_1 + cx_2 \end{bmatrix}$$

$$= ax_1^2 + bx_1x_2 + bx_1x_2 + cx_2^2 = ax_1^2 + 2bx_1x_2 + cx_2^2$$

$$ax_1^2 + 2bx_1x_2 + cx_2^2 = x^t\begin{bmatrix} a & b \\ b & c \end{bmatrix}x, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

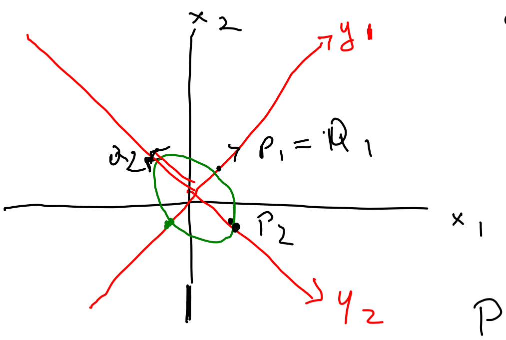
$$X = PY = y_1P_1 + y_2P_2 \qquad Y = P^{-1}X = P^TX$$

$$2x_1^2 + 2x_1x_2 + 2x_2^2 = Y^TP^TAPY = Y^T\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}Y = 3y_1^2 + y_2^2$$

Therefore we can identify for example the curve $2x_1^2 + 2x_1x_2 + 2x_2^2 = 2$ by an orthogonal change of coordinates. In the coordinate system whose basis is P_1, P_2 the the curve has equation $3y_1^2 + y_2^2 = 1$

$$\text{i.e.}\quad \frac{y_1^2}{\left(\frac{1}{\sqrt{3}}\right)^2} + \frac{y_2^2}{1^2} = 1$$

ellipse with major axis along P_2 and minor axis along P_1



Length of major axis $= 2$

" " minor axis $= \frac{2}{\sqrt{3}}$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad \det(P) = -1$$

choose $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ instead of P here

$$Q = \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix} \qquad \text{rotation matrix}$$

$$P^T A P = Q^T A Q$$

$$x = PY = QZ \qquad Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad z_1 = y_1,\ z_2 = -y_2$$

$$= y_1 P_1 + y_2 P_2 = z_1 Q_1 + z_2 Q_2$$

Another application

Find the maximum and minimum values of $2x_1^2 + 2x_1x_2 + x_2^2$ subject to the constraint

$$x_1^2 + x_2^2 = 1$$

$$2x_1^2 + 2x_1x_2 + x_2^2 = 3y_1^2 + y_2^2 \quad \text{and} \quad y_1^2 + y_2^2 = 1$$

$$x_1^2 + x_2^2 = [x_1, x_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [y_1, y_2]\underbrace{P^TP}_{= I}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y_1^2 + y_2^2$$

$$1 = y_1^2 + y_2^2 \le 3y_1^2 + y_2^2 \le 3y_1^2 + 3y_2^2 = 3(y_1^2 + y_2^2) = 3$$

$\therefore$ the max. & min value are $3, 1$

and they occur at $y_1 = \pm 1, y_2 = 0$ i.e. $\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

and $y_1 = 0, y_2 = \pm 1$ i.e. $\pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ resp.

General Case:

$$q(x_1, \dots, x_n) = a_{11}x_1^2 + \dots + a_{nn}x_n^2 + \sum_{i \ne j} a_{ij}x_ix_j$$

$$\boxed{a_{ij} = a_{ji}} \qquad = \sum_{i=1}^n a_{ii}x_i^2 + 2\sum_{i<j} a_{ij}x_ix_j$$

$$= X^TAX \qquad X = [x_1, \dots, x_n]^T$$

$$A = [a_{ij}].$$

q quadratic form in n variables

A is its matrix with resp to standard basis

$$[x_1, \dots, x_n]\begin{bmatrix} a_{11} & \dots a_{1n} \\ & \\ a_{n1} & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [x_1, \dots, x_n]\begin{bmatrix} a_{11}x_1 + \dots + a_{1n}x_n \\ \\ a_{n1}x_1 + \dots + a_{nn}x_n \end{bmatrix}$$

$$= q(x_1, \dots, x_n)$$

$x_1^2 + 2x_1x_2 + x_2x_3 + x_3^2$

$$= [x_1, x_2, x_3] \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1/2 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1/2 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}$$ matrix of the quadratic form w.r.t the standard basis

Indication of the proof of the theorem that any symmetric matrix is orthogonally diagonalizable.

Want P with $P^{-1} = P^T$ & $P^{-1}AP$ a diagonal matrix.

Step I. Let $AX = \mu X$, $AY = \lambda Y$ $\quad X, Y \neq 0$

want to prove $X \cdot Y = X^T Y = 0$

$$\mu X \cdot Y = AX \cdot Y = (AX)^T Y = X^T A^T Y = X^T A Y = \lambda X^T Y$$

$$\Rightarrow \mu X \cdot Y = \lambda X \cdot Y \Rightarrow (\mu - \lambda) X \cdot Y = 0$$

$$\Rightarrow X \cdot Y = 0 \text{ since } \lambda \neq \mu$$

Step II: The eigenvalues of A are real.

Complex numbers are of the form $a + ib$ where $a, b \in \mathbb{R}$, $i^2 = -1$

$(a+ib)(c+id) = (ac-bd) + i(ad+bc)$

$\overline{(a+ib)} = a - ib$ complex conjugate

$z = a + ib$ real $\Leftrightarrow b = 0$

$\Leftrightarrow z = \bar{z}$

$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

$z\bar{z} = |z|^2 = a^2 + b^2$

Let $AX = \lambda X$, $\lambda \in \mathbb{C}$

$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\bar{X} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix}$

$$\bar{\lambda} \bar{X} \cdot X = \overline{AX} \cdot X = \overline{AX}^T X = \bar{X}^T \bar{A}^T X$$

$$= \bar{X}^T A X = \lambda \bar{X}^T X$$

$\Rightarrow$ jj.

$$(\bar{\lambda} - \lambda) \underbrace{\bar{X}^T X} = 0$$

$\bar{X}^T X \neq 0$
$= |x_1|^2 + \cdots + |x_n|^2$

$$\Rightarrow \lambda = \bar{\lambda}$$

Step III: A is diagonalizable

Let W = subspace of $\mathbb{R}^n$ spanned by the eigenvectors of A. We want to show $W = \mathbb{R}^n$ or, equivalently $W^\perp = 0$. If $W^\perp \neq 0$, let $u_1, \dots, u_p$ be an orthogonal basis for $W^\perp$. If $v \in W^\perp$ we claim that $Av \in W^\perp$. Indeed, if $w \in W$ then

$$Av \cdot w = (Av)^T w = v^T A^T w = v^T A w = 0$$

since $w \in W \Rightarrow Aw \in W$. If $Bu_j = b_{1j} u_1 + \dots + b_{pj} u_p$ and $v = y_1 u_1 + \dots + y_p u_p$ then $Av = z_1 u_1 + \dots + z_p u_p$. If $Z = [z_1, \dots, z_p]^T$, $Y = [y_1, \dots, y_p]^T$, $B = [b_{ij}]$ then $Z = BY$. The matrix B is symmetric since for any $u = s_1 u_1 + \dots + s_p u_p \in W$ we have $(Au)^T v = u^T A v$ and $(Au)^T v = (BS)^T Y = S^T B^T Y$, $u^T A v = S^T B Y$. So $S^T B Y = S^T B^T Y$ for any $Y, S \in \mathbb{R}^p$ which $\Rightarrow B = B^T$. Let $BZ = \lambda Z$ with $Z \neq 0$. Then $Av = \lambda v$ so that $v \in W \cap W^\perp$ and $v \neq 0$ which cannot happen. So $W^\perp = 0$.

Step IV. $\mathbb{R}^n$ has an orthog basis of eigenvectors of A

To find such a basis find an orthogonal basis of each eigenspace. By step I the resulting set of vectors will be orthogonal and they span $\mathbb{R}^n$ by step III.

Example: If $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, $E_4 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$, $E_1 = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ is an orthog basis of E_1. So $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ is an orthog basis of eigenvectors of A.

The matrix $P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$ is orthogonal

and $P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P^TAP$ since $P^T = P^{-1}$

Since $2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 = X^TAX$

and $X^TAX = Y^T \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} Y$ if $X = PY$

$= 4y_1^2 + y_2^2 + y_3^2$

we have

$$2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 = 4y_1^2 + y_2^2 + y_3^2$$

so that the surface $X^TAX = 1$ is an ellipsoid. Since

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2 \le 4y_1^2 + y_2^2 + y_3^2 \le 4(y_1^2 + y_2^2 + y_3^2) = 4(x_1^2 + x_2^2 + x_3^2)$$

we see that the maximum and minumum values of X^TAX subject to the constraint $x_1^2 + x_2^2 + x_3^2 = 1$ are 4 and 1 and that these values are attained at $(1,1,1)$ and $(1/\sqrt{2}, -\frac{1}{\sqrt{2}}, 0)$. As an exercise, find all places where the max and min are attained subject to the constraint $x_1^2 + x_2^2 + x_3^2 = 1$.