Gram-Schmidt Process

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Let $u_1, \dots, u_m \in \mathbb{R}^n$, linearly independent
Convert this set of vectors into an orthogonal set
$v_1, v_2, \dots, v_m$ in such a way that
$W_i = \text{span}(u_1, \dots, u_i) = \text{span}(v_1, \dots, v_i)$

$$v_1 = u_1$$
$$v_2 = u_2 - \text{proj}_{W_1}(u_2) = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$$
$$v_3 = u_3 - \text{proj}_{W_2}(u_3) = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2$$
$$\vdots$$
$$v_i = u_i - \text{proj}_{W_{i-1}}(u_i) = u_i - \frac{u_i \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_i \cdot v_2}{v_2 \cdot v_2} v_2 - \cdots - \frac{u_i \cdot v_{i-1}}{v_{i-1} \cdot v_{i-1}} v_{i-1}$$

Example. $u_1 = (1, -1, 0, 0)$, $u_2 = (1, 0, -1, 0)$, $u_3 = (1, 0, 0, -1)$
$W = \text{span}(u_1, u_2, u_3)$

$$v_1 = u_1 = (1, -1, 0, 0)$$
$$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (1, 0, -1, 0) - \frac{1}{2}(1, -1, 0, 0)$$
$$= \cancel{\tfrac{1}{2}}(1, 1, -2, 0)$$
$$v_3 = u_3 - \frac{u_3 \cdot v_1}{v \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2$$
$$= (1, 0, 0, -1) - \frac{1}{2}(1, -1, 0, 0) - \frac{1}{6}(1, 1, -2, 0)$$
$$= \frac{1}{6}(2, 2, 2, -6) = \frac{1}{3}(1, 1, 1, -3)$$

$\therefore \underbrace{(1, -1, 0, 0)}_{v_1}, \underbrace{(1, 1, -2, 0)}_{v_2}, \underbrace{(1, 1, 1, -3)}_{v_3}$ orthog basis for W

$v = (1, 2, 3, 4)$ orthog of v onto W

$$\text{proj}_W(v) = \frac{v \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{v \cdot v_2}{v_2 \cdot v_2} v_2 + \frac{v \cdot v_3}{v_3 \cdot v_3} v_3$$
$$= \frac{-1}{2}(1, -1, 0, 0) + \frac{-3}{6}(1, 1, -2, 0) + \frac{-6}{12}(1, 1, 1, -3)$$
$$= -\frac{1}{2}(3, 1, -1, -3)$$

$\operatorname{perp}_W(v) = v - \operatorname{proj}_W(v) = \operatorname{proj}_{W^\perp}(v)$

$= (1,2,3,4) + \frac{1}{2}(3,1,-1,-3)$

$= \left(\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}\right) = \frac{5}{2}(1,1,1,1)$

$W^\perp = \operatorname{span}((1,1,1,1)) = \operatorname{null}([1,1,1,1])$

= solution set of the equation $x_1 + x_2 + x_3 + x_4 = 0$

Example 2. $W = \operatorname{span}(\underset{u_1}{(1,1,-1,0)}, \underset{u_2}{(1,1,1,1)})$

Compute $W^\perp$.

$v_1 = u_1$, $v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$

$= (1,1,1,1) - \frac{1}{3}(1,1,-1,0) = \frac{1}{3}(2,2,4,3)$

$\underset{v_1}{(1,1,-1,0)}, \underset{v_2}{(2,2,4,3)}$ orthog basis for W

$\underset{v_1}{(1,1,-1,0)}, \underset{v_2}{(2,2,4,3)}, \underset{u_1}{(1,0,0,0)}, \underset{u_2}{(0,1,0,0)}$ basis for $\mathbb{R}^4$

$v_3 = u_1 - \operatorname{proj}_W(u_1) = (1,0,0,0) - \frac{1}{3}(1,1,-1,0) - \frac{2}{33}(2,2,4,3)$

$= \frac{1}{33}(18,-15,3,-6)$

$v_4 = u_2 - \operatorname{proj}_W(u_2) = (0,1,0,0) - \frac{1}{3}(1,1,-1,0) - \frac{2}{33}(2,2,4,3)$

$= \frac{1}{33}(-15,18,3,-6)$

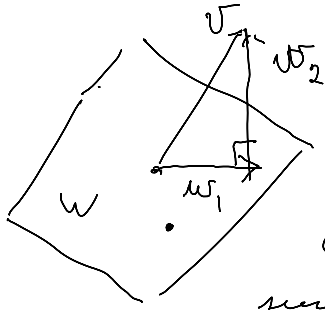
$W^\perp = \operatorname{span}((18,-15,3,-6), (-15,18,3,-6))$

$W = (W^\perp)^\perp \Rightarrow W$ solution space of the system

$-15x_1 + 18x_2 + 3x_3 - 6x_4 = 0$

$18x_1 - 15x_2 + 3x_3 - 6x_4 = 0$

Geometric characterization of the orthogonal projection of a vector v on a subspace W



$w_1 = \mathrm{proj}_W(v)$, $w_2 = \mathrm{perp}_W(v) = v - \mathrm{proj}_W(v)$

$v = w_1 + w_2$ $\quad w_1 \in W$, $w_1 \cdot w_2 = 0$

Claim w_1 is the unique vector of W such that $\|v - w\|$ is smallest possible

Indeed, if $w \in W$ then

$$\|v - w\|^2 = \|\underbrace{v - w_1}_{w_2} + \underbrace{w_1 - w}_{u}\|^2 = \|w_2\|^2 + \|u\|^2$$

$$w_2 \cdot u = 0$$

Theorem (Pythagoras) $v_1 \cdot v_2 = 0 \Leftrightarrow \|v_1 + v_2\|^2 = \|v_1\|^2 + \|v_2\|^2$

Pf. $\|v_1 + v_2\|^2$

$= (v_1 + v_2) \cdot (v_1 + v_2)$ $\qquad v^2 = v \cdot v$

$= v_1^2 + 2 v_1 \cdot v_2 + v_2^2$

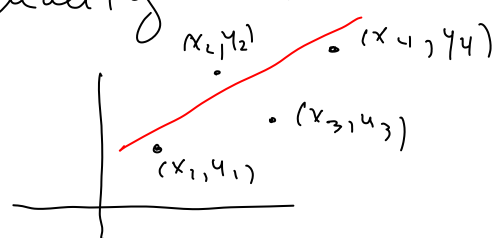
$= \|v_1\|^2 + 2 v_1 \cdot v_2 + \|v_2\|^2$

$= \|v_1\|^2 + \|v_2\|^2 \iff v_1 \cdot v_2 = 0$ Q.E.D

$$\|v - w\|^2 = \|w_2\|^2 + \|w_1 - w\|^2 \geq \|w_2\|^2$$

with equality $\iff w = w_1$

Application



given 4 pts (x_i, y_i) in the plane

find the line of best fit

$y = mx + b$ equation of a line in $\mathbb{R}^2$

This line passes through the given pts $\Leftrightarrow$

$$mx_1 + b = y_1$$
$$mx_2 + b = y_2$$
$$mx_3 + b = y_3$$
$$mx_4 + b = y_4$$

$$\Leftrightarrow \quad \underbrace{(y_1, y_2, y_3, y_4)}_{Y} = m\underbrace{(x_1, x_2, x_3, x_4)}_{X} + b\underbrace{(1,1,1,1)}_{U}$$

If no line passes through the given points we want to find the line that 'best fits' the data. This means that we want to make

$$\|Y - mX - bU\| \quad \text{smallest possible.}$$

$\therefore$ Want $mX + bU = \text{proj}_W(Y)$, $W = \text{span}(X, U)$

Then solve for m, b to get the equation of the line of best fit..