1

Orthogonality in $\mathbb{R}^n$

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Let W be a subspace of $\mathbb{R}^n$

$W^\perp$ = orthogonal complement of W in $\mathbb{R}^n$

$= \{ v \in \mathbb{R}^n \mid v \cdot w = 0 \text{ for all } w \in W \}$

$W^\perp$ subspace of $\mathbb{R}^n$: $v_1, v_2 \in W^\perp$, $a, b \in \mathbb{R}$

$w \cdot (a v_1 + b v_2) = a\, w \cdot v_1 + b\, w \cdot v_2 = 0$ for $w \in W$

$0 \in W^\perp$ since $0 \cdot w$ for any $w \in \mathbb{R}^n$.

If $W = \text{span}(v_1, \ldots, v_m)$ then

$W^\perp = \{ v \in \mathbb{R}^n \mid v \cdot v_1 = v \cdot v_2 = \cdots = v \cdot v_m = 0 \}$

since $v(a_1 v_1 + \cdots + a_m v_m) = a_1\, v \cdot v_1 + \cdots + a_n\, v \cdot v_m$

$= 0$ if $v \cdot v_i = 0$ for $1 \le i \le m$

Ex. $W = \text{span}([1,-1,0], [1,0,-1])$

$W^\perp = \{ (x,y,z) \mid x - y = 0,\ x - z = 0 \}$

$= \text{span}((1,1,1))$

$(W^\perp)^\perp = \{ (x,y,z) \mid x + y + z = 0 \}$

$= \text{span}((1,-1,0), (0,1,-1))$

$\dim W = m \Rightarrow \dim W^\perp = n - m$

Let $W = \text{span}(v_1, \ldots, v_m)$ + suppose $v_1, \ldots, v_m$ basis for W.

and let A = $m \times n$ matrix whose rows are the vectors $v_1, \ldots, v_m$. Then $m = \text{rank}(A)$. But

$W^\perp = \{ v \in \mathbb{R}^n \mid v_i \cdot v = 0 \text{ for } 1 \le i \le m \} = \text{null}(A)$

$\Rightarrow \dim W^\perp = n - \text{rank}(A) = n - m$.

Ex. $W = \text{span}((1,-1,0), (0,1,-1))$. $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

$\text{null}(A) = \{ (x,y,z) \mid x - y = 0,\ y - z = 0 \} = \text{span}((1,1,1))$

$(W^\perp)^\perp = W$: $\underset{n-(n-m) = m}{(W^\perp)^\perp} \supseteq \underset{m}{W}$ but here same dimension

$\Rightarrow (W^\perp)^\perp = W$

$W^\perp \cap W = \{0\}$ since $v \in W \cap W^\perp \Rightarrow v \cdot v = 0$

$\|v\|^2 = 0 \Rightarrow v = 0$

$\mathbb{R}^n = W + W^\perp$ i.e. any $v \in \mathbb{R}^n$ can be written as a sum of a vector in W and a vector in $W^\perp$:

$v \in \mathbb{R}^n \Rightarrow v = w_1 + w_2$, $w_1 \in W, w_2 \in W^\perp$

Let $u_1, \dots, u_m$ be a basis for W

$v_1, \dots, v_k$ " " " " $W^\perp$ $(k = n-m)$

Claim: $u_1, \dots, u_m, v_1, \dots, v_k$ lin. indep. This would imply that they span $\mathbb{R}^n$.

$$\underbrace{a_1u_1 + \dots + a_mu_m}_{w_1 \in W} + \underbrace{b_1v_1 + \dots + b_kv_k}_{w_2 \in W^\perp} = 0$$

$w_1 + w_2 = 0 \Rightarrow w_1 = -w_2 \Rightarrow w_1 \in W \cap W^\perp$

$\Rightarrow w_1 = 0 = w_2$

$\Rightarrow a_1u_1 + \dots + a_mu_m = 0$, $b_1v_1 + \dots + b_kv_k = 0$

$\Rightarrow a_1 = \dots = a_m = b_1 = \dots = b_k = 0$

$\therefore u_1, \dots, u_m, v_1, \dots, v_k$ span $\mathbb{R}^n$ (n lin. indep vectors in $\mathbb{R}^n$ span $\mathbb{R}^n$)

$$\therefore v \in \mathbb{R}^n \Rightarrow v = \underbrace{a_1u_1 + \dots + a_mu_m}_{w_1 \in W} + \underbrace{b_1v_1 + \dots + b_kv_k}_{w_2 \in W^\perp}$$

if $v = w_1 + w_2 = w_1' + w_2'$ where $w_1' \in W$, $w_2' \in W^\perp$

then $\underset{\in W}{w_1 - w_1'} = \underset{\in W^\perp}{w_2' - w_2} \Rightarrow w_1 - w_1' \in W \cap W^\perp$

$\Rightarrow w_1 - w_1' = 0$

$\Rightarrow w_1 = w_1' \Rightarrow w_2 = w_2'$

Let $v \in \mathbb{R}^n$. Then there are unique vectors $w_1 \in W$, $w_2 \in W^\perp$ such that

$$v = w_1 + w_2$$

$w_1 = \text{proj}_W(v) =$ orthog proj of v onto W

$w_2 = \text{proj}_{W^\perp}(v) =$ orthog proj of v onto $W^\perp$

$= \text{perp}_W(v) =$ proj of v perpendicular to W

<u>Explicit Formulas</u>:

Suppose $W = \text{span}(v_1, .., v_m)$ $\quad v_1, .., v_m$ non-zero & orthog. i.e. $v_i \cdot v_j = 0$ for $i \neq j$

$v \in W \Rightarrow v = a_1 v_1 + .. + a_m v_m$

$$v \cdot v_i = a_i v_i \cdot v_i \Rightarrow a_i = \frac{v \cdot v_i}{v_i \cdot v_i}$$

$$\Rightarrow \quad v = \frac{v \cdot v_1}{v_1 \cdot v_1} v_1 + \cdots + \frac{v \cdot v_m}{v_m \cdot v_m} v_m$$

$0 = a_1 v_1 + \cdots + a_m v_m \Rightarrow a_i = 0$ for all i

$\Rightarrow$ non-zero orthog vectors are lin. indep.

Let $v \in \mathbb{R}^n$. Let $w_1 = \frac{v \cdot v_1}{v_1 \cdot v_1} v_1 + \cdots + \frac{v \cdot v_m}{v_m \cdot v_m} v_m \in W$

Claim $w_2 = v - w_1 \perp W$ since

$$w_2 \cdot v_i = v \cdot v_i - \frac{v \cdot v_i}{\cancel{v_i \cdot v_i}} \cancel{v_i \cdot v_i} = 0$$

$\therefore \quad w_1 \in W, \; w_2 \in W^\perp$

$$\Rightarrow \quad \text{proj}_W(v) = \frac{v \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{v \cdot v_2}{v_2 \cdot v_2} v_2 + \cdots + \frac{v \cdot v_m}{v_m \cdot v_m} v_m$$

Example: $W = \text{span}([1,1,0,0], [0,0,1,1])$
$\qquad\qquad\qquad\qquad\qquad\qquad v_1 \qquad\qquad v_2$

$v_1 \cdot v_2 = 0$

$v = (1,2,3,4) \in \mathbb{R}^4$

$$\text{proj}_W(v) = \frac{v \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{v \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= \frac{3}{2}(1,1,0,0) + \frac{7}{2}(0,0,1,1) = \frac{1}{2}(3,3,7,7)$$

$$\text{perp}_W(v) = v - \text{proj}_W(v)$$
$$= (1,2,3,4) - \frac{1}{2}(3,3,7,7)$$
$$= \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$$