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A, B $n\times n$ matrices

A similar to B if there is an invertible matrix P with $P^{-1}AP = B \iff A = Q^{-1}AQ$ where $Q = P^{-1}$. So A similar to $B \iff B$ similar to A.

$$\det(A-\lambda I) = \det(B-\lambda I) \quad \text{if} \quad B = P^{-1}AP$$

$\therefore$ Similar matrices have same char. polyn.

$$\det(P^{-1}AP) = \cancel{\det}(P)^{-1}\det(A)\cancel{\det}(P) = \det(A)$$

$$tr(P^{-1}AP) = tr(APP^{-1}) = tr(A)$$

$$\det(A-\lambda I) = (-1)^n(\lambda^n + a_1\lambda^{n-1} + \cdots + a_n) = (-1)^n \Delta_A(\lambda)$$

$$\Delta_A(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = \text{normalized char. polyn of } A$$

Will show that the a_i's can be expressed in terms of subdeterminants of A.

Case $n=2$: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\Delta_\lambda(A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{vmatrix} = (\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21}$$

$$= \lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$$

$$= \lambda^2 - tr(A)\lambda + \det(A)$$

$$= (\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2$$

$\Rightarrow \operatorname{tr}(A) = \lambda_1 + \lambda_2$, $\det(A) = \lambda_1 \lambda_2$

Example $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \Rightarrow \Delta_A(\lambda) = \lambda^2 - 4\lambda + 1$

roots of char poly are $\dfrac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$

$\lambda_1 = 2 + \sqrt{3}$, $\lambda_2 = 2 - \sqrt{3}$ eigenvalues of A

$\lambda_1 + \lambda_2 = 4$, $\lambda_1 \lambda_2 = 4 - 3 = 1$

~~P~~Problem. Construct a 2×2 matrix A with eigenvalues $2+\sqrt{2}$, $2-\sqrt{2}$

Solution. $\operatorname{tr}(A) = 4$, $\det(A) = 2$

$A = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$ works , $B = \begin{bmatrix} 4 & 2 \\ -1 & 0 \end{bmatrix}$ also works

A and B are similar ~~m~~matrices since

$P^{-1}AP = \begin{bmatrix} 2+\sqrt{2} & 0 \\ 0 & 2-\sqrt{2} \end{bmatrix} = D$ $\quad P = [P_1, P_2]$

$AP_1 = (2+\sqrt{2})P_1$, $AP_2 = (2-\sqrt{2})P_2$

$Q^{-1}BQ = \begin{bmatrix} 2+\sqrt{2} & 0 \\ 0 & 2-\sqrt{2} \end{bmatrix} = D$ $\quad Q = [Q_1, Q_2]$

$BQ_1 = (2+\sqrt{2})Q_1$, $BQ_2 = (2-\sqrt{2})Q_2$

$A = PDP^{-1} = PQ^{-1}BQP^{-1} = R^{-1}BR$, $R = QP^{-1}$

$\Rightarrow$ A similar to B

Case $(n=3)$ $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\Delta_A(\lambda) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{vmatrix} = (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})$$

$$- a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}$$

$$- a_{13}a_{31}(\lambda - a_{22})$$

$$- a_{32}a_{23}(\lambda - a_{11}) - (\lambda - a_{33})a_{21}a_{12}$$

$$= \lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + \left(\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \right)\lambda$$

$$- \det(A)$$

<u>Ex</u> $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ $\qquad \Delta_A(\lambda) = \lambda^3 - 6\lambda^2 + 9\lambda - 4$

General formula for $\Delta_A(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$

$$a_i = (-1)^i \, tr_i(A)$$

where $tr_i(A)$ = sum of the principal $i \times i$ minors of A

principal minor of A = det of the submatrix det. by rows $r_1, r_2, \dots, r_i$ and cols $r_1, r_2, \dots, r_i$

$r_1 < r_2 < \dots < r_i$

$tr_1(A) = tr(A)$, $tr_n(A) = \det(A)$

$$tr_i(A) = \sum_{k_1 < \dots < k_i} \lambda_{k_1} \cdots \lambda_{k_i}$$

Ex. $n = 3$ $\qquad tr_2(A) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$

We now show that a symmetric 2×2 matrix is diagonalizable.

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$$\Delta_A(\lambda) = \lambda^2 - (a+c)\lambda + ac - b^2$$

Discriminant of this polyn is $(a+c)^2 - 4(ac-b^2)$
$= (a+c)^2 - 4ac + 4b^2 = (a-c)^2 + 4b^2 \geq 0$

roots of char polyn. $= \dfrac{a+c \pm \sqrt{(a-c)^2+4b^2}}{2}$

Let λ_1, λ_2 be the eigenvalues of A
Let P_1 be an eigenvector with eigenvalue λ_1
If $\lambda_1 \neq \lambda$ let P_2 be an eigenvector with eigenvalue λ_2
Claim $P_1 \cdot P_2 = 0 = P_1^T P_2$

$$\lambda_1 P_1 \cdot P_2 = AP_1 \cdot P_2 = (AP_1)^T P_2 = P_1^T A P_2 = P_1^T \lambda_2 P_2$$
$$= \lambda_2 P_1^T P_2$$

$\Rightarrow \lambda_1 P_1 \cdot P_2 = \lambda_2 P_1 \cdot P_2$

$\Rightarrow (\lambda_1 - \lambda_2) P_1 \cdot P_2 = 0 \quad \Rightarrow P_1 \cdot P_2 = 0$ since $\lambda_1 \neq \lambda_2$

Now suppose $\lambda_1 = \lambda_2$.
Let P_2 be any col matrix orthogonal to P_1
claim AP_2 is orthogonal to P_1
Indeed, $AP_2 \cdot P_1 = (AP_2)^T P_1 = P_2 \cdot AP_1 = P_2 \cdot (\lambda_1 P_1)$
$= \lambda_1 P_2 \cdot P_1 = 0$
$\Rightarrow AP_2 = c P_2 \Rightarrow P$ is an eigenvector of A & c is an eigenvalue $\Rightarrow c = \lambda_1$
as λ_1 is the only eigen value.

$P = [P_1, P_2] \qquad Q = [Q_1, Q_2]$

$Q_i = P_i / \|P_i\|$ Now Q has for columns an orthogonal basis of $\mathbb{R}^2$ consisting of vectors of unit length, i.e. $Q^T Q = I$

$$P^T P = \begin{bmatrix} \|P_1\|^2 & 0 \\ 0 & \|P_2\|^2 \end{bmatrix}$$

$\Rightarrow Q^{-1} = Q^T$ such a matrix called orthogonal

To do the above in general we need to work in $\mathbb{R}^n$ with the standard inner product

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$$

$$X \cdot Y = x_1 y_1 + \cdots + x_n y_n = X^T Y$$

$$\|X\| = \sqrt{X \cdot X} = \sqrt{x_1^2 + \cdots + x_n^2}$$

X, Y orthogonal if $X \cdot Y = 0$

$P_1, \dots, P_k \in \mathbb{R}^n$ called orthogonal vectors if $P_i \cdot P_j = 0$ for $i \neq j$

If in addition $\|P_i\| = 1$ for all i the vectors are called orthonormal e.g. standard basis is orthonormal

Ex $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$ orthogonal vectors

$\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ orthonormal set of vectors

$p_1, p_2, \dots, p_n$ are orthonormal vectors in $\mathbb{R}^{n \times 1}$

$\Longleftrightarrow \quad P^T P = I \qquad$ where $P = [p_1, p_2, \dots, p_n]$

since (i,j) entry of $P^T P = p_i^T p_j = p_i \cdot p_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$\Longleftrightarrow \quad P$ invertible & $P^{-1} = P^T$