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A $n\times n$ matrix $\Rightarrow$ char polyn of A

$$\det(A-\lambda \underset{\sim}{I}) = \pm(\lambda^n + a_1\lambda^{n-1} + \cdots + a_n)$$

roots of this polyn are the eigenvalues of A

Ex. $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2-\lambda & 1 \end{vmatrix}$$

$$= (2-\lambda)(4-4\lambda+\lambda^2-1) - (1-\lambda) + (-1+\lambda)$$

$$= (2-\lambda)(\lambda^2-4\lambda+3) - 2 + 2\lambda$$

$$= 2\lambda^2 - 8\lambda + 6 - \lambda^3 + 4\lambda^2 - 3\lambda - 2 + 2\lambda$$

$$= -\lambda^3 + 6\lambda^2 - 9\lambda + 4$$

$$= -(\lambda^3 - 6\lambda^2 + 9\lambda - 4) = f(\lambda)$$

If c is an integer root then $c \mid 4$

$\therefore\ c = \pm 1, \pm 2, \pm 4$

$f(1) = 0 \quad \therefore\ 1$ is a root $\Leftrightarrow \lambda - 1 \mid f(\lambda)$

$$\begin{array}{r l}
 & \lambda^2 - 5\lambda + 4 \\
\lambda - 1 & \overline{)\ \lambda^3 - 6\lambda^2 + 9\lambda - 4} \\
 & \underline{\lambda^3 - \lambda^2} \\
 & -5\lambda^2 + 9\lambda \\
 & \underline{-5\lambda^2 + 5\lambda} \\
 & 4\lambda - 4 \\
 & \underline{4\lambda - 4} \\
 & 0
\end{array}$$

$$\therefore (\lambda^3 - 6\lambda^2 + 9\lambda - 4)$$

$$= (\lambda-1)(\lambda^2 - 5\lambda + 4)$$

$$= (\lambda-1)(\lambda-1)(\lambda-4)$$

$$= (\lambda-1)^2(\lambda-4)$$

Let $\Delta(\lambda)$ = char. polyn of A, $\Delta(c) = 0$

$$\Delta(\lambda) = (\lambda - c)^m g(\lambda) \qquad g(c) \neq 0$$
$$\iff \lambda - c \nmid g(\lambda)$$

Def: m = algebraic multiplicity of the eigenvalue c

$\dim(E_c) = \dim \operatorname{null}(A - cI)$
= geometric multiplicity of the eigenvalue c

Fact: Geometric mult $\leq$ alg. mult.

Ex. $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ 1, 4 eigenvalues

$$\Delta(\lambda) = -(\lambda-1)^2(\lambda-4)$$

alg. mult of 1 is 2 ; alg mult of 4 is 1

$E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$ $\dim E_1 = 2$ = geom. mult of 1

$E_4 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$ $\dim E_4 = 1$ = geom mult of 4

$P_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, P_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, P_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

eigenvectors of A and are lin. indep.
$\therefore$ they are a basis for $\mathbb{R}^3$

if $P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ then $AP = PD$ $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$\iff P^{-1}AP = D$ since P invertible

$\iff A = PDP^{-1}$

$\Rightarrow \quad A^n = PD^nP^{-1} \qquad D^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4^n \end{bmatrix}$

Find B with $B^2 = A$

Let $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Then $C^2 = D$

Let $B = PCP^{-1}$. Then $B^2 = PC^2P^{-1}$
$= PDP^{-1} = A$

$A^{-1} = PD^{-1}P^{-1}$
since $A = PDP^{-1}$

$D = \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{bmatrix} \qquad f(\lambda) = a_0\lambda^k + \cdots + a_k$

$f(D) = \begin{bmatrix} f(c_1) & 0 & \cdots & 0 \\ 0 & f(c_2) & \cdots & 0 \\ 0 & 0 & & f(c_n) \end{bmatrix}$

Ex $\quad A = P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}P^{-1} \quad \Rightarrow \quad f(A) = a_0A^k + \cdots + a_kI$
$= P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & & f(4) \end{bmatrix}P^{-1}$

$e^A = P\begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e^4 \end{bmatrix}P^{-1} \qquad f(\lambda) = 1 + \lambda + \frac{\lambda^2}{2} + \cdots + \frac{\lambda^n}{n!} + \cdots$

$e^A = I + A + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \cdots$

If $\Delta(\lambda) = (\lambda - c_1)^{m_1}(\lambda - c_2)^{m_2} \cdots (\lambda - c_k)^{m_k}$
$c_1, \ldots, c_k$ distinct

then A diagonalizable $\Leftrightarrow$ geom. mult of c_i
$= m_i$ for all i

If $P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = D$ then $A = P \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} P^{-1}$

$$\Delta(\lambda) = \det(A - \lambda I) = \det(PDP^{-1} - \lambda I)$$
$$= \det(PDP^{-1} - \lambda P P^{-1} I)$$
$$= \det(P(D - \lambda I)P^{-1}) = \cancel{\det(P)} \det(D - \lambda I) \cancel{\det(P^{-1})}$$
$$= \det(D - \lambda I) \qquad \text{since } \det(P^{-1}) = \det(P)^{-1}$$
$$= \begin{vmatrix} \lambda_1 - \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda_n - \lambda \end{vmatrix} = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$$

Ex. of a non-diagonalizable matrix

$A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \qquad \begin{vmatrix} -\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = -\lambda + \lambda^2 + 1$ char polyn

$= \lambda^2 - \lambda + 1$

has no real roots

$\therefore$ A not diagonalizable.

Ex. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2$ char polyn.

$\therefore$ 0 only eigenvalue $\quad \dim E_0 = \dim \text{null}(A) = 1$

$\neq 2 = $ alg mult

$E_0 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

A diagonalizable $\iff$ there is an invertible matrix P with $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$\iff$ P invertible and $AP = PD$

$\iff$ $AP_i = \lambda_i P_i$ & $P_1, \ldots, P_n$ lin indep ($P_i = i$-th col of P)

Theorem If $P_1, \ldots, P_k$ are eigenvectors of A with distinct eigenvalues λ_i then $P_1, \ldots, P_k$ are linearly independent

Proof:
$$\underbrace{a_1 P_1}_{Q_1} + \cdots + \underbrace{a_k P_k}_{Q_k} = 0$$

$$a_i = 0 \iff Q_i = 0 \quad \text{since} \quad P_i \neq 0$$

$$(1) \qquad Q_1 + \cdots + Q_k = 0 \qquad AQ_i = \lambda_i Q_i$$

$$AQ_1 + \cdots + AQ_k = 0$$

$$(2) \qquad \lambda_1 Q_1 + \cdots + \lambda_k Q_k = 0$$

$$(2) - \lambda_1 (1): \quad \underbrace{(\lambda_2 - \lambda_1) Q_2}_{R_2} + \cdots + \underbrace{(\lambda_k - \lambda_1) Q_k}_{R_k} = 0$$

$$R_2 + \cdots + R_k = 0 \qquad AR_i = \lambda_i R_i$$

Repeat this process $k-2$ times to get

$$(\lambda_k - \lambda_1)(\lambda_k - \lambda_2) \cdots (\lambda_k - \lambda_{k-1}) Q_k = 0$$

$$\Rightarrow Q_k = 0 \quad \text{since} \quad \lambda_k \neq \lambda_i \text{ for } k \neq i$$

$$\Rightarrow Q_1 + \cdots + Q_{k-1} = 0$$

Apply above $k-1$ times to get $Q_1 = Q_2 = \quad = Q_k = 0$

Cor If P is an eigenvector with eigenvalue λ then P cannot be written as a linear comb of eigenvectors with eigenvalues $\neq \lambda$.