Audio recording started: 10:08 AM Thursday, November 06, 2003

$$\begin{vmatrix} x & y & z & 1 \\ a_1 & b_1 & c_1 & 1 \\ a_2 & b_2 & c_2 & 1 \\ a_3 & b_3 & c_3 & 1 \end{vmatrix} = 0$$

Equation of the plane passing through the points $A_1(a_1, b_1, c_1)$, $A_2(a_2, b_2, c_2)$, $A_3(a_3, b_3, c_3)$ if these points are not collinear

$$\parallel$$

$$\begin{vmatrix} x-a_3 & y-b_3 & z-c_3 & 0 \\ a_1-a_3 & b_1-b_3 & c_1-c_3 & 0 \\ a_2-a_3 & b_2-b_3 & c_2-c_3 & 0 \\ a_3 & b_3 & c_3 & 1 \end{vmatrix} = \begin{vmatrix} x-a_3 & y-b_3 & z-c_3 \\ a_1-a_3 & b_1-b_3 & c_1-c_3 \\ a_2-a_3 & b_2-b_3 & c_2-c_3 \end{vmatrix}$$

$$= (x-a_3, y-b_3, z-c_3) \cdot \overrightarrow{A_1A_3} \times \overrightarrow{A_2A_3}$$

Vandermonde determinant

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & & & \vdots \\ 1 & x_n & x_3^2 & \cdots & x_n^{n-1} \end{vmatrix} = \begin{matrix} (x_n - x_{n-1})(x_n - x_{n-2}) \cdots (x_n - x_1) \\ (x_{n-1} - x_{n-2}) \cdots (x_{n-1} - x_1) \\ \parallel \qquad \vdots \\ V_n(x_1, x_2, \ldots, x_n) \qquad (x_2 - x_1) \end{matrix}$$

$\neq 0 \iff x_1, \ldots, x_n$ are all distinct

Ex $n=4$

$$\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & x_2 - x_1 & x_2^2 - x_1x_2 & x_2^3 - x_1x_2^2 \\ 1 & x_3 - x_1 & x_3^2 - x_1x_3 & x_3^3 - x_1x_3^2 \\ 1 & x_4 - x_1 & x_4^2 - x_1x_4 & x_4^3 - x_1x_4^2 \end{vmatrix}$$

$$(x_4 - x_1)(x_3 - x_1)(x_2 - x_1) \quad V_3(x_2, x_3, x_4)$$

Vandermonde determinant appears in interpolation problem. eg. Find the curve $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ passing through the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ (x_4, y_4) (x_1, x_2, x_3, x_4 distinct)

$$a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 = y_1$$
$$a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3 = y_2$$
$$a_0 + a_1 x_3 + a_2 x_3^2 + a_3 x_3^3 = y_3$$
$$a_0 + a_1 x_4 + a_2 x_4^2 + a_3 x_4^3 = y_4$$

det of coeff. matrix $= V_1(x_1, x_2, x_3, x_4) \neq 0$

$\therefore$ system has a unique sol.

Cramer's Rule : $AX = B$ $\quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

$A = [A_1, A_2, \dots, A_n]$ $\quad A_i$ = i-th col of A

eq. to

$$x_1 A_1 + x_2 A_2 + \dots + x_n A_n = B$$

<u>Claim</u> : $\det(A) \neq 0$ then there is a unique sol.

$$x_i = \frac{\det(A_1, \dots, B, \dots, A_n)}{\det(A_1, \dots, A_i, \dots, A_n)}$$

i-th col

Proof :

$$\det(A_1, \dots, B, \dots, A_n) = \det\left(A_1, \dots, B - \sum_{j \neq i} x_j A_j, \dots, A_n\right)$$
$$= \det(A_1, \dots, x_i A_i, \dots, A_n) = x_i \cdot \det(A)$$

QED.

λ eigenvalue of the $n \times n$ matrix $A = [a_{ij}]$

$\iff$ there is a non-zero vector X with $AX = \lambda X$

$\iff$ $(A - \lambda I)X = 0$ has a non-trivial sol

$\iff$ $\det(A - \lambda I) = 0$

$\det(A - \lambda I)$ '' polynomial of degree n called the characteristic polynomial of A

the roots of this polynomial are the eigenvalues of A.

Ex. $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & \lambda-1 & 1-\lambda \end{vmatrix}$$

$$= (\lambda-1)\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & -1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} 2-\lambda & 1 & 2 \\ 1 & 2-\lambda & 3-\lambda \\ 0 & 1 & 0 \end{vmatrix}$$

$$= -(\lambda-1)\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = -(\lambda-1)(\lambda^2 - 5\lambda + 6 - 2)$$

$$= -(\lambda-1)^2(\lambda-4)$$

$\therefore$ 1, 4 eigenvalues

1 has alg. mult. 2

4 '' '' '' 1

E_1 = eigenspace for eigenv. 1 $= Null\left(\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{A-I}\right) = Span\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$

Geometric multiplicity of eigenvalue 1 $= \dim E_1 = 2$

E_4 = eigenspace for eign 4 $= Null\left(\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\right) = Span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$

Geometric mult of eigenv 4 $= \dim E_4 = 1$

Claim: $P_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, P_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, P_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ lin indep and $\therefore$ a basis for $\mathbb{R}^3$

Compute A^n: Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$

$$X = y_1 P_1 + y_2 P_2 + y_3 P_3 = PY, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\therefore Y = P^{-1} X \qquad P = [P_1, P_2, P_3]$$

$$AX = y_1 P_1 + y_2 P_2 + 4 y_3 P_3$$

$$A^n X = y_1 P_1 + y_2 P_2 + 4^n y_3 P_3$$

$$AX = P \begin{bmatrix} y_1 \\ y_2 \\ 4y_3 \end{bmatrix} = PDY \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= PDP^{-1} X$$

$$\Rightarrow A = PDP^{-1}$$

$$\Rightarrow A^n = PD^n P^{-1}$$

Def. A, C similar matrices if $C = PAP^{-1}$ for some invertible matrix P.

If A similar to a diagonal matrix then we say A is diagonalizable.

An $n \times n$ matrix A is diagonalizable $\Leftrightarrow$ there is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \qquad P^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{vmatrix} = 3$$

$$A^n = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4^n \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 4^n & 4^n & 4^n \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2+4^n & -1+4^n & -1+4^n \\ -1+4^n & 2+4^n & -1+4^n \\ -1+4^n & -1+4^n & 2+4^n \end{bmatrix}$$