Lecture18

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$A = [a_{ij}]$ $n \times n$ matrix

$$\det(A) = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \sum_{\substack{\text{permutation} \\ \text{of } 1,2,\ldots,n}} \operatorname{sign}(j_1,\ldots,j_n)\, a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

$n!$ terms

$\operatorname{sign}(j_1,\ldots,j_n) = (-1)^{\text{parity}(j_1,\ldots,j_n)}$

parity = even or odd depending on whether or not an even or odd number of transpositions is required to transform $1, 2, \ldots, n$ to $j_1, j_2, \ldots, j_n$

Ex.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right. = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

DANGER

NO SUCH SIMPLE FORMULA EXISTS for $n > 3$

Properties of Determinants

1. $\det(A^T) = \det(A)$

Notation: $\det(A) = \det(A_1, \ldots, A_n)$ A_i = i-th row of A

2. If $A_i = A_j$ with $i \neq j$ then $\det(A) = 0$

3. $\det(A_1, \ldots, A_i, \ldots, A_j, \ldots, A_n) = -\det(A_1, \ldots, A_j, \ldots, A_i, \ldots, A_n)$
   $i \neq j$

④ $\det(A_1, \ldots, \underset{A_i}{\underset{\shortparallel}{aX+bY}}, \ldots, A_n) = a \det(A_1, \ldots, X, \ldots, A_n)$

$+ b \det(A_1, \ldots, Y, \ldots, A_n)$

Eg. $\det(A_1, \ldots, cA_i, \ldots, A_n) = c \det(A_1, \ldots, A_i, \ldots, A_n)$

Ex $\begin{vmatrix} 3\cdot 1 & 3\cdot 4 & 3\cdot 6 \\ 3\cdot 2 & 3\cdot 5 & 3\cdot 7 \\ 3\cdot 1 & 3\cdot 1 & 3\cdot 8 \end{vmatrix} = 3^3 \begin{vmatrix} 1 & 4 & 6 \\ 2 & 5 & 7 \\ 1 & 1 & 8 \end{vmatrix}$

⑤ $\det(A_1, \ldots, A_i + aA_j, \ldots, A_n)$

$j \neq i$

$= \det(A_1, \ldots, A_i, \ldots, A_n) + a \cancel{\det(A_1, \ldots, A_j, \ldots, A_n)}$

$= \det(A)$

⑥ $\det(I) = 1$

Also have $\det \begin{bmatrix} a_1 & & * \\ & a_2 & \\ 0 & & \ddots \; a_n \end{bmatrix} = a_1 a_2 \cdots a_n$

Ex $\begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 0 & -3 & -5 \\ 0 & -3 & -7 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 0 & -3 & -5 \\ 0 & 0 & -2 \end{vmatrix} = 6$

Effect of elementary row operations on the det

(a) Interchange 2 rows: det changes sign

(b) Mult a row by a scalar c: det gets mult. by c

(c) Add a multiple of a row to another row: det does not change

$$I \xrightarrow{\xi} E \qquad A \xrightarrow{\xi} B$$

(a) ξ : Interchange two rows

$$\det(E) = -1 \,, \quad \det(B) = -\det(A) = \det(E)\det(A)$$

(b) ξ : Mult a row by c

$$\det(E) = c \,, \quad \det(B) = c\det(A) = \det(E)\det(A)$$

(c) ξ : Add a mult of a row to another row.

$$\det(E) = 1 \,, \quad \det(B) = \det(A) = \det(E)\det(A)$$

<u>Claim</u> $\det(AB) = \det(A)\det(B)$

Case 1. A, B are invertible

$$A = E_1 \cdots E_k \,, \; B = F_1 \cdots F_\ell \Rightarrow AB = E_1 \cdots E_k F_1 \cdots F_\ell$$

E_i, F_j elementary matrices

$$\Rightarrow \det(AB) = \det(E_1)\det(E_2)\cdots\det(E_k)\det(F_1)\det(F_2)\cdots\det(F_\ell).$$
$$= \det(A)\det(B)$$

Case 2. Suppose one of A, B not invertible.
If A not invertible then $\det(A) = 0$
claim AB not invertible; otherwise, there is a matrix D with $ABD = I$ which implies that A is invertible which is a contradiction. To see the last assertion, note that if $CA = I$ then $AX = 0 \Rightarrow X = 0$
$\therefore$ if A has a left inverse then A is invertible
taking transposes can show that $AD = I$
$\Rightarrow A^T$ invertible $\Rightarrow A$ invertible

A invertible $\Rightarrow$ A row-equiv to I $\Rightarrow$ $\det(A) \neq 0$

A not invertible $\Rightarrow$ A row equiv to a matrix with a zero row $\Rightarrow$ $\det(A) = 0$

$\therefore$ A invertible $\Leftrightarrow$ $\det(A) \neq 0$

Minors and Cofactors of $A = [a_{ij}]$

m_{ij} = (i,j)-th minor of A = determinant of $(n-1)\times(n-1)$ matrix obtained from A by deleting i-th row and j-th col.

$c_{ij} = (-1)^{i+j} m_{ij}$ = (i,j)-th cofactor of A

$C = [c_{ij}]$ = cofactor matrix of A

C^T = adjoint of A = $\operatorname{adj}(A)$

Laplace expansion formulae

① $\det(A) = a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots + a_{in}c_{in}$ expansion along i-th row

② $\det(A) = a_{1j}c_{1j} + a_{2j}c_{2j} + \cdots + a_{nj}c_{nj}$ expansion along j-th column

Example

$$\begin{vmatrix} 1 & 2 & 0 & 4 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 3 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{vmatrix} = -2\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

Example of computing an adjoint

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

$$c_{11} = \begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix} = -1, \quad c_{12} = -\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = -4, \quad c_{13} = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 5$$

$$c_{21} = -\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = -2, \quad c_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2, \quad c_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = -2$$

$$c_{31} = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1, \quad c_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0, \quad c_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$$

$$C = [c_{ij}] = \begin{bmatrix} -1 & -4 & 5 \\ -2 & 2 & -2 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\operatorname{adj}(A) = C^T = \begin{bmatrix} 1 & -2 & 1 \\ -4 & 2 & 0 \\ 5 & -2 & -1 \end{bmatrix}$$

$$A \operatorname{adj}(A) = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -4 & 2 & 0 \\ 5 & -2 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = -2I$$

$$\det(A) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -2$$

General fact: $A \operatorname{adj}(A) = \operatorname{adj}(A) A = \det(A) I$

$\therefore \det(A) \neq 0 \Rightarrow A$ invertible with inverse $\frac{1}{\det(A)} \operatorname{adj}(A)$.

Exercise. Prove $A \operatorname{adj}(A) = A \operatorname{adj}(A) = \det(A) I$ for $n = 3$

General case proved in the same way: uses Laplace expansion formulas.