Audio recording started: 10:15 AM Thursday, October 30, 2003

A scalar λ is an eigenvalue of an $n\times n$ matrix A if there is a non-zero $X\in\mathbb{R}^n$ such that

$$AX = \lambda X = \lambda I X$$

$$\iff \quad (A-\lambda I)X = 0$$

$$E_\lambda = \{X \mid (A-\lambda I)X = 0\} = \text{Null space of } A-\lambda I$$

Let $X \in$ Null space of $A-\lambda I$ are the eigenvectors of A for the eigenvalue λ

E_λ = eigenspace of A for the eigenvalue λ

Ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, eigenvalues $= \{3, -1\}$

$$E_3 = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} = \text{Null space of } \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$E_{-1} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} = \text{Null space of } \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

λ eigenvalue $\iff A-\lambda I$ not invertible

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A-\lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$

and $A-\lambda I$ not invertible $\iff \det(A-\lambda I) = 0$

i.e. $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

i.e. $\lambda^2 - (a+d)\lambda + ad - bc = 0$

$\lambda^2 - Tr(A)\lambda + \det(A) = 0$, $Tr\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+d$

Want to define the determinant of an $n\times n$ matrix

Case $n = 3$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = [A_1, A_2\, A_3]$$

$$A_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ a_{3i} \end{bmatrix} = i\text{-th col of } A$$

<u>Solve</u> $\quad AX = B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

ie $\quad x_1 A_1 + x_2 A_2 + x_3 A_3 = B$

dot both sides with

$$A_2 \times A_3 = \begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \end{bmatrix}$$

get $\quad x_1 A_1 \cdot A_2 \times A_3 + \cancel{x_2 A_2 \cdot A_2 \times A_3} + \cancel{x_3 A_3 \cdot A_2 \times A_3} = B \cdot A_2 \times A_3$

$\Rightarrow \quad x_1 A_1 \cdot A_2 \times A_3 = B \cdot A_2 \times A_3$

since $A_2 \cdot A_2 \times A_3 = A_3 \cdot A_2 \times A_3 = 0$

$\Rightarrow \quad x_1 = \dfrac{B \cdot A_2 \times A_3}{A_1 \cdot A_2 \times A_3}$ if $A_1 \cdot A_2 \times A_3 \neq 0$

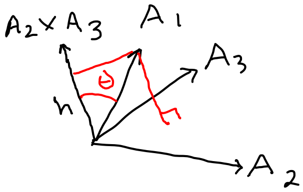
$$A_1 \cdot A_2 \times A_3 = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= \det(A)$$

$A_1 \cdot A_2 \times A_3$ is known as the triple scalar product of the vectors A_1, A_2, A_3.

$$|A_1 \cdot A_2 \times A_3| = \|A_1\| \|A_2 \times A_3\| \cos\theta$$

where θ = angle between A_1 and $A_2 \times A_3$



$h = \|A_1\| \cos\theta$ = magnitude of orthogonal projection of A_1 on $A_2 \times A_3$

$\therefore$ h = height of box with sides // to A_1, A_2, A_3. Since $\|A_2 \times A_3\|$ = area of //gm with sides // to A_2, A_3 we see that

$|A_1 \cdot A_2 \times A_3|$ = volume of box.

Continuing with our computation of x_1, x_2, x_3 we dot both sides of $x_1 A_1 + x_2 A_2 + x_3 A_3 = B$ with $A_1 \times A_3$ and $A_1 \times A_2$ to get

$$x_2 A_2 \cdot A_1 \times A_3 = B \cdot A_1 \times A_3, \quad x_3 A_3 \cdot A_1 \times A_2 = B \cdot A_1 \times A_2$$

<u>Claim</u>: $A_2 \cdot A_1 \times A_3 = -A_1 \cdot A_2 \times A_3$

This implies $A_3 \cdot A_1 \times A_2 = -A_1 \cdot A_3 \times A_2 = A_1 \cdot A_2 \times A_3$

since $A_3 \times A_2 = -A_2 \times A_3$.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31}$$

There are 6 terms of the form $\pm a_{1j_1} a_{2j_2} a_{3j_3}$, one for each permutation of 1 2 3, where the sign is + if $j_1 j_2 j_3$ can be obtained from 1 2 3 by an number

number of interchanges and − if an odd number of interchanges is required. Thus

$$\det(A) = \sum \operatorname{sign}(j_1 j_2 j_3)\, a_{1j_1} a_{2j_2} a_{3j_3}$$

where the summation is over all permutations $j_1 j_2 j_3$ of 123. This definition can be extended to give the definition of $\det(A)$ for an $n \times n$ matrix $A = [a_{ij}]$:

$$\det(A) = \sum \operatorname{sign}(j_1 j_2 \cdots j_n)\, a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

where the summation is over all permutations of $12 \cdots n$. There are $n!$ terms in the sum.

Now let's prove that the determinant changes sign if two columns are interchanged. We do it for $n = 3$ and where columns 1 and 2 are interchanged. The general proof is done in the same way.

$$\begin{vmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{vmatrix} = a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31}$$
$$a_{13}a_{22}a_{31} - a_{11}a_{22}a_{33}$$
$$+ a_{11}a_{23}a_{32} - a_{13}a_{21}a_{32}$$
$$= -\det(A)$$

This $\Rightarrow$ $\det(A) = 0$ if two columns are equal. One can also show that $\det(A) = \det(A^T)$ and that $\det(A) = a_{1j}c_{1j} + a_{2j}c_{2j} + \cdots + a_{nj}c_{nj}$ where $c_{ij} = (-1)^{i+j} \times$ determinant of $(n-1)\times(n-1)$ matrix obtained from A by deleting i-th row and j-th column $=$ (i,j)-th cofactor of A. Without the sign it is called the (i,j)-th minor of A.