Audio recording started: 10:06 AM Tuesday, October 28, 2003

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ which assigns to each vector u in $\mathbb{R}^n$ a vector $T(u)$ in $\mathbb{R}^m$ is said to be a linear transformation if

$$T(au + bv) = aT(u) + bT(v)$$

Example If A is an $m \times n$ matrix then you can define a linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ as follows:

$$T_A(X) = AX$$

where we write the vectors of $\mathbb{R}^n$ and $\mathbb{R}^m$ as column matrices. It is linear since

$$T_A(aX + bY) = A(aX + bY) = aAX + bAY$$
$$= aT_A(X) + bT_A(Y)$$

Example (1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$ $\quad X \in \mathbb{R}^3$

$$T_A(X) = AX = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + 3z \\ x + z \end{bmatrix} \in \mathbb{R}^2$$

$$T_A(x, y) = (x + 2y + 3z, \ x + z)$$

(2) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $\quad T_A : \mathbb{R}^2 \to \mathbb{R}^2$

$$T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix}$$

If $T:\mathbb{R}^n \to \mathbb{R}^m$, $S:\mathbb{R}^m \to \mathbb{R}^p$ are linear then their composition $S\circ T$ defined by

$$S\circ T(x) = S(T(x))$$

$S\circ T$ is linear since

$$\begin{aligned} S\circ T(au+bv) &= S(T(au+bv)) \\ &= S(aT(u)+bT(v)) \\ &= a\,S(T(u)) + b\,S(T(v)) \\ &= a\,S\circ T(u) + b\,S\circ T(v) \end{aligned}$$

If $T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_B : \mathbb{R}^m \to \mathbb{R}^p$

A $m\times n$ $\qquad$ B $p\times m$

$$T_A(X) = AX, \quad T_B(Y) = BY$$

$$\begin{aligned} T_B\circ T_A(X) &= T_B(T_A(X)) = B(AX) = (BA)X \\ &= T_{BA} \end{aligned}$$

Any linear transf of $\mathbb{R}^n$ into $\mathbb{R}^m$ is of the form T_A for some $m\times n$ matrix A. Indeed if $X\in\mathbb{R}^n$ can written

$$x = x_1e_1 + \cdots + x_ne_n \qquad e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

so $T(X) = x_1T(e_1) + \cdots + x_nT(e_n)$

if we let $T(e_i) = A_i$ and let A be

the $m \times n$ matrix whose columns are $A_1, \ldots, A_m$ then $T(x) = AX$, i.e. $T = T_A$. The matrix A is called the standard matrix of T. If $u_1, \ldots, u_n$ is a basis for $\mathbb{R}^n$ then $X \in \mathbb{R}^n$ can be written

$$X = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$$

where $a_1, \ldots, a_n$ are uniquely determined by X (these are the coord. of X w.r.t. $u_1, \ldots, u_n$)

$$T(x) = a_1 T(u_1) + a_2 T(u_2) + \cdots + a_n T(u_n)$$

(w.r.t = with respect to) So T is completely determined by its effect on a basis

Example: Given a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ with $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $T\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ find the standard matrix of T.

$$\begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \begin{aligned} a + b &= x \\ 2a + 3b &= y \end{aligned}$$

$$b = y - 2x, \quad a = x - b = 3x - y$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\therefore T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 3T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) - 2T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$$

$$= 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(-\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}1\\3\end{bmatrix}\right)$$

$$= -T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) + T\left(\begin{bmatrix}1\\3\end{bmatrix}\right)$$

$$= -\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

Therefor the standard matrix of T is

$$A = \begin{bmatrix}-1 & 1\\1 & 0\end{bmatrix}$$

i.e. $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}-1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-x+y\\x\end{bmatrix}$

$$T(x,y) = (-x+y, x)$$

If $u_1, \dots, u_n$ basis for $\mathbb{R}^n$ and $v_1, \dots, v_n$ are vectors in $\mathbb{R}^m$ then there is a unique linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(u_i) = v_i$. Indeed, if $u = a_1u_1 + \dots + a_nu_n$ then define $T(u) = a_1v_1 + \dots + a_nv_n$. Then $T(u_i) = v_i$ and T is linear for if

$$v = b_1u_1 + \dots + b_nu_n$$

then $T(au+bv) = T\left((aa_1+bb_1)u_1 + \dots + (aa_n+bb_n)u_n\right)$

$$= (aa_1+bb_1)v_1 + \dots + (aa_n+bb_n)v_n$$

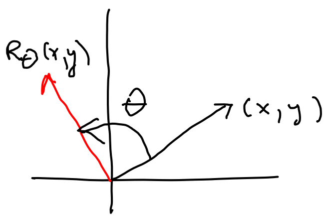
$$= a(a_1v_1 + \dots a_nv_n) + b(b_1v_1 + \dots + b_nv_n)$$

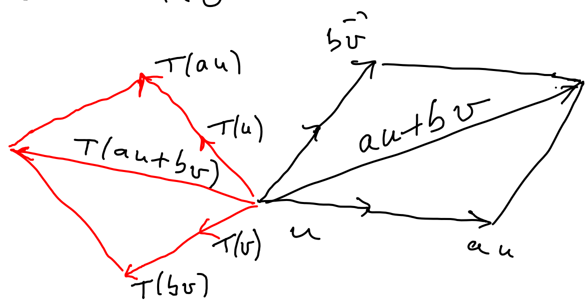
$$= aT(u) + bT(v)$$

Geometric Examples of Linear Transformations

① Rotation about the origin in the plane $\mathbb{R}^2$ clockwise through an angle θ

Claim R_θ linear





$$\Rightarrow R_\theta(au+bv) = R_\theta(au) + R_\theta(bv) = aR_\theta(u) + bR_\theta(v)$$

$$R_\theta\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}, \quad R_\theta\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

$\therefore$ standard matrix of $R_\theta = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

$$R_\theta\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\cos\theta - y\sin\theta\\ x\sin\theta + y\cos\theta\end{bmatrix}$$

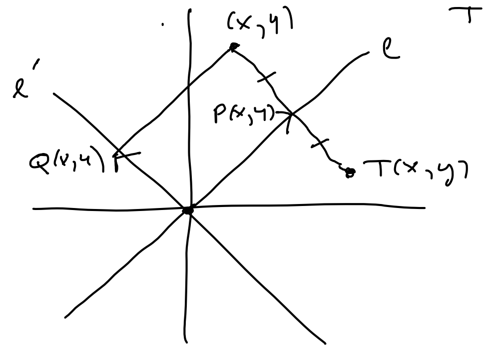
$$R_\theta(x,y) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$$

$$R_{\theta_1+\theta_2} = R_{\theta_1}\circ R_{\theta_2} = R_{\theta_2}R_{\theta_1}$$

$$\begin{bmatrix}\cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2)\\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2)\end{bmatrix} = \begin{bmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{bmatrix}\begin{bmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{bmatrix}$$

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$
$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$$

② Reflection in the line $\ell : ax+by=0$



$T(x,y)$ = reflection of (x,y) in ℓ

Let $P(x,y)$ = orthogonal projection of (x,y) on ℓ. Then

$$P(x,y) = (x,y)\cdot\frac{(-b,a)}{a^2+b^2}(-b,a)$$

$$= \left(\frac{b^2}{a^2+b^2}x - \frac{ab}{a^2+b^2}y,\ \frac{-ab}{a^2+b^2}x + \frac{a^2}{a^2+b^2}y\right)$$

$\Rightarrow$ P linear

Let Q = orthog. projection on the line $\ell' : -bx+ay=0$

Then $Q(x,y) = (x,y)\cdot\frac{(a,b)}{a^2+b^2}(a,b)$

$(x,y) = P(x,y)+Q(x,y) \Rightarrow P(x,y) = (x,y)-Q(x,y)$

$\Rightarrow T(x,y) = (x,y) - 2Q(x,y) = (x,y) - 2\,(x,y)\cdot\frac{(a,b)}{a^2+b^2}(a,b)$

<u>Exercise</u>. Show T linear and find standard matrix of T

Example: $\ell : x-y=0$

$$T(x,y) = (x,y) - 2(x,y)\cdot\frac{(1,-1)}{2}(1,-1)$$

$$= (x,y) - (x-y,\ y-x) = (y,x)$$

<u>Def.</u> If $T:\mathbb{R}^n \to \mathbb{R}^n$ is linear then a non-zero vector $u\in\mathbb{R}^n$ is said to be an eigenvector of T if $T(u) = cu$ for some scalar c. The scalar c is called the eigenvalue of the eigenvector u.

Example Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Then $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an eigenvector of T if

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = c\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{for some scalar } c$$

ie $$\begin{bmatrix} x+2y \\ 2x+y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$$

or equivalently, $$\begin{aligned} x+2y &= cx \\ 2x+y &= cy \end{aligned}$$

which we can write $$\begin{aligned} (1-c)x + 2y &= 0 \\ 2x + (1-c)y &= 0 \end{aligned}$$

Since (x,y) is a non-zero solution the coefficient matrix is not invertible and so its determinant

$$\begin{vmatrix} 1-c & 2 \\ 2 & 1-c \end{vmatrix} = 0$$

$\Rightarrow (1-c)^2 - 4 = 0 \Rightarrow c^2 - 2c - 3 = 0 \Rightarrow (c-3)(c+1) = 0$

$\Rightarrow c = -1$ or 3.

if $c = -1$ then $(x,y) = a(1,-1)$ and

if $c = 3$ then $(x,y) = a(1,1)$

$\Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ eigenvector of T with eigenvalue -1

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ eigenvector of T with eigenvalue 3