Linearity

We begin with a problem illustrating the use of linearity to solve it

Audio recording started: 10:24 AM Thursday, October 23, 2003

Problem. Find the plane containing the line of intersection of the planes $x+y+2z=1$ and $2x+3y+z=2$ which passes through the point $(1,1,1)$.

Solution. $A(x+y+2z)+B(2x+3y+z)=A+2B$

lin comb of the two given planes.

$$(1)\quad (A+2B)x+(A+3B)y+(2A+B)z=A+2B$$

$$(A+2B,\ A+3B,\ 2A+B)=A(1,1,2)+B(2,3,1)$$
$$\neq 0 \quad \text{if } A,B\neq 0$$

The plane (1) contains the line of intersection of the given planes and we want it to pass through $(1,1,1)$.

$$(A+2B)(1)+(A+3B)(1)+(2A+B)(1)=A+2B$$

$$3A+4B=0 \qquad \text{choose } A=-4,\ B=3$$

We can use this to find a plane which does not meet at the same time both of the given planes. For example, adding the equations and increasing the constant by 1 we get $3x+4y+3z=4$ which works since it is a translate of a plane passing through the line of intersection of the given planes to a distinct parallel plane. Hence it does not meet the line of intersection of the two given planes.

Prolem. Find a formula for the n-th term of the sequence

$$\underset{x_0}{1},\ \underset{x_1}{1},\ \underset{x_2}{-1},\ \underset{x_3}{-11},\ \underset{x_4}{-49},\ \ldots$$

where the rule of formation of the sequence is $x_{n+2} = 5x_{n+1} - 6x_n$, $x_1 = x_2 = 1$.

Let $A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$. Then

$$A\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A\begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ -6x_n + 5x_{n+1} \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix}$$

$$A\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix},$$

$$A^2\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}, \quad A^n\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = A^n\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} -6 & 5 \\ -30 & 19 \end{bmatrix}$$

The matrix A has a very nice property

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

<u>Def</u> A col. matrix $X \neq 0$ such that $AX = cX$ is called an eigenvector of A, the scalar c called the eigenvalue of the eigenvector X.

$$\begin{bmatrix} x \\ y \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{since } \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ basis for } \mathbb{R}^2$$

$$a + b = x \qquad b = y - 2x$$
$$2a + 3b = y \qquad a = x - b = 3x - y$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = (3x - y)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (y - 2x)\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$A^n\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3A^n\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2A^n\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$= 3\cdot 2^n\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2\cdot 3^n\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 3\cdot 2^n - 2\cdot 3^n \\ 3\cdot 2^{n+1} - 2\cdot 3^{n+1} \end{bmatrix}$$

$$A^n\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -A^n\begin{bmatrix} 1 \\ 2 \end{bmatrix} + A^n\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$= -2^n\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3^n\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} -2^n + 3^n \\ -2^{n+1} + 3^{n+1} \end{bmatrix}$$

$$\therefore A^n = \begin{bmatrix} 3\cdot 2^n - 2\cdot 3^n & -2^n + 3^n \\ 3\cdot 2^{n+1} - 2\cdot 3^{n+1} & -2^{n+1} + 3^{n+1} \end{bmatrix}$$

$$\begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = A^n\begin{bmatrix} 1 \\ 1 \end{bmatrix} = A^n\left(2\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$$
$$= 2A^n\begin{bmatrix} 1 \\ 2 \end{bmatrix} - A^n\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 2^{n+1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3^n\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2^{n+1} - 3^n \\ 2^{n+2} - 3^{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$$

$\therefore$ $x_n = 2^{n+1} - 3^n$ gives a formula for the n-th term of the given sequence. For example, $x_4 = 2^5 - 3^4 = 32 - 81 = -49$