Basis and Dimension

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10:36 AM

Let S be a subspace of $\mathbb{R}^n$ and let $u_1, \ldots, u_m$ be a basis for S. Since $u_1, \ldots, u_m$ span S, every $v \in S$ can be written in the form

$$(1) \qquad v = a_1 u_1 + \cdots + a_m u_m$$

If $v = b_1 u_1 + \cdots + b_m u_m$ then subtracting from (1) we get

$$0 = (a_1 - b_1) u_1 + \cdots + (a_m - b_m) u_m$$

which implies $a_i - b_i = 0$, i.e. $a_i = b_i$, for $1 \le i \le m$ since $u_1, \ldots, u_m$ are linearly independent. Thus every vector $v \in S$ can be uniquely written in the form (1). The m-tuple $(a_1, \ldots, a_m)$ is called the coordinate vector of $v \in S$ with respect to the basis $u_1, \ldots, u_m$.

Example. Let $S = \{(x, y, z) \mid x + y + z = 0\}$, a plane passing through 0. Then $u = (x, y, z) \in S$ $\iff u = (-y-z, y, z) = y(-1, 1, 0) + z(-1, 0, 1)$. The vectors $u_1 = (-1, 1, 0)$, $u_2 = (-1, 0, 1)$ are in S and form a basis for S since they span S and are linearly independent. Thus $\dim(S) = 2$ since this basis has two vectors in it. This agrees with our intuition that a plane is 2-dimensional. Now $\mathbb{R}^3$ is 3-dimensional with basis $(1,0,0), (0,1,0), (0,0,1)$ and $u = (x,y,z) \in S$ has coordinate vector (x,y,z) with respect to this basis of $\mathbb{R}^3$ but it also has the coordinate vector (y,z) with respect to the basis u_1, u_2. The vector (y,z) is enough information to locate the vector $u \in S$.

The following is an important property of the dimension of a subspace.

Theorem. Let S be an m-dimensional subspace of $\mathbb{R}^n$ and let $u_1, \dots, u_m \in S$. Then $u_1, \dots, u_m \in S$ is a basis for S if either $u_1, \dots, u_m$ are linearly independent or $u_1, \dots, u_m$ span S.

Proof. If $u_1, \dots, u_m$ are linearly independent and $S \neq \text{span}(u_1, \dots, u_m)$ then there is a vector $u \in S$ with $u \notin \text{span}(u_1, \dots, u_m)$. But then $u_1, \dots, u_m, u$ would be a independent set of $m+1$ vectors in an m-dimensional subspace which is impossible. If $u_1, \dots, u_m$ span S and are not independent then one of the u_i is a linear combination of the other u_j and so S would be generated by $m-1$ vectors which is impossible.

Let us now treat the problem of finding a basis for a subspace S of $\mathbb{R}^n$ where

$$S = \text{span}(u_1, u_2, \dots, u_m)$$

We give two methods. Both use Gauss-Jordan reduction. The second method gives more information

Method I. Write the vectors $u_1, \dots, u_m$ as the rows of a matrix A and row reduce A to echelon form B. Then the non-zero rows of B are a basis for S.

Since S = row space of A, this follows from the fact that the row space of A is unchanged by elementary row operations and the fact that the non-zero rows of a matrix in echelon form are linearly independent.

Example. Let $S = \text{span}((1,2,1),(2,1,2),(1,5,1))$
Then S = rowspace of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 5 & 1 \end{bmatrix}$$

Row reducing A to echelon form, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 5 & 1 \end{bmatrix} \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - R_1}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 0 & 3 & 0 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $(1,2,1)$, $(0,-3,0)$ are a basis for S and S is 2-dimensional.

Method II. Write the given vectors as columns of a matrix A and row reduce A to echelon form. Then the pivot columns of A are a basis for S = column space of A.

Example. $S = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}\right)$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - R_1}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} = B$$ and the

pivot columns are columns 1 and 2 so that columns 1 and 2 of A are a basis for S. Note that the column space of A is not equal to the column space of B since $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ does not belong to the column space of B. Thus row operations on a matrix can change the column space.

We now explain why method II works. Let A be an $n \times m$ matrix and let $A_1, A_2, \ldots, A_m$ be the columns of A. We want to find the dependence relations

$$x_1 A_1 + x_2 A_2 + \cdots + x_m A_m = 0$$

among the column vectors $A_1, \ldots, A_m$. This is equivalent to finding all solutions $X = [x_1, x_2, \ldots, x_m]^T$ of the system of equations $AX = 0$. To find these solutions, we row reduce A to reduced echelon form B. If $j_1, j_2, \ldots, j_r$ are the pivot columns, then the pivot variables $x_{j_1}, \ldots, x_{j_r}$ can be expressed as linear combinations of the non-pivot variables $x_{i_1}, \ldots, x_{i_s}$. Then $AX = 0 \iff$

$$X = x_{i_1} v_1 + \cdots + x_{i_s} v_s$$

where v_k is the solution of $AX = 0$ with $x_{i_k} = 1$ and $x_{i_l} = 0$ for $l \neq k$. The vectors $v_1, \ldots, v_s$ span the solution space of $AX = 0$ and they are linearly independent since the i_k-th component of

$x_{i_1}v_1 + \cdots + x_{i_s}v_s$ is x_{i_k}. Since there is a dependence relation $x_1u_1 + \cdots + x_m u_m = 0$ with $x_{i_k} = 1$ and $x_{i_l} = 0$ for $l \neq k$ we see that $u_{i_k} \in \text{span}(u_{j_1}, \ldots, u_{j_r})$. To show that $u_{j_1}, \ldots, u_{j_r}$ are linearly independent we simply note that any dependence relation $c_{j_1}u_{j_1} + \cdots + c_{j_r}u_{j_r} = 0$ can be written as the dependence relation $x_1u_1 + \cdots + x_m u_m = 0$

where $x_{j_i} = c_{j_i}$ for $1 \leq i \leq r$ and $x_{i_k} = 0$ for $1 \leq k \leq s$. But this implies $x_{j_i} = c_{j_i} = 0$ since the independent variables x_{j_i} are all zero.

QED

The above shows that the dimension of the row space of a matrix A is equal to the dimension of the column space of A. The common value is called the rank of A and is denoted by $\text{rank}(A)$. If $r = \text{rank}(A)$ and m is the number of columns of A then the dimension of the solution space of $AX = 0$ is $m - r$. This subspace is called the null space of A. Its dimension is called the nullity of A and is denoted by $\text{nullity}(A)$.

Theorem (Rank Theorem) If A is a matrix then

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = \#\text{ of cols of } A$$

Example.

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 0 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 - R_1 \\ R_4 - R_1}]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_4 - R_3} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

$$AX = 0 \iff x_1 = -2x_2 + x_4,\ x_3 = -x_4$$

$$\iff (x_1, x_2, x_3, x_4) = (-2x_2 + x_4,\ x_2,\ -x_4,\ x_4)$$

$$= x_2(-2, 1, 0, 0) + x_4(1, 0, -1, 1)$$

$\therefore$ $(-2,1,0,0)$, $(1,0,-1,1)$ basis for nullspace

$(1,2,0,-1)$, $(0,0,1,1)$ basis for row space

$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}$ (cols 1, 3 of A) basis for columnspace

$\operatorname{rank}(A) = 2$, $\operatorname{nullity}(A) = 2$