Matrices

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Matrix operations

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

i = row index
j = column index

$a_{ij} = (i,j)$-th entry of A

size of $A = (m,n)$ also say A is $m \times n$

$\mathbb{R}^{m\times n}$ = set of all $m \times n$ matrices

$$\mathbb{R}^{m\times 1} = \left\{ \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \middle| \; a_{i1} \in \mathbb{R} \right\}$$ set of col matrices

$$\mathbb{R}^{1\times n} = \left\{ [a_{11}, a_{12}, \ldots, a_{1n}] \mid a_{1j} \in \mathbb{R} \right\}$$
set of row matrices

Addition of matrices

$A = [a_{ij}]$, $B = [b_{ij}] \in \mathbb{R}^{m\times n}$

$A + B = [c_{ij}]$ where $c_{ij} = a_{ij} + b_{ij}$

Multiplication by scalars

$c \in \mathbb{R}$, $A = [a_{ij}] \in \mathbb{R}^{m \times n}$

$$cA = [c a_{ij}]$$

Ex $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$

$$2 \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 2 & 2 \end{bmatrix}$$

Vector properties hold

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$0 \in \mathbb{R}^{m \times n}$ zero matrix $\quad 0 = [a_{ij}] \quad a_{ij} = 0$ for all i, j

$A = [a_{ij}]$, $-A = [-a_{ij}]$

$$A + (-A) = (-A) + A = 0$$

Matrix multiplication

$$\underset{1 \times n}{[x_1, x_2, \ldots, x_n]} \underset{n \times 1}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}} = \underset{1 \times 1}{[x_1 y_1 + \cdots + x_n y_n]}$$

$$= x_1 y_1 + \cdots + x_n y_n$$

$A = [a_{ij}]$ $m \times n$ $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $n \times 1$

$$\underset{m \times 1}{AX} = \begin{bmatrix} A_1 X \\ A_2 X \\ \vdots \\ A_m X \end{bmatrix}$$ where A_i = i-th row of A

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$A = [a_{ij}]$ $m \times n$, $B = (b_{ij})$ $n \times p$

j-th column of AB = $A \times$ j-th column of B

$\therefore (i,j)$-th entry of AB is

$$[a_{i1}, a_{i2}, \ldots, a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$ = i-th row of A $\times$ j-th col of B

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$
$$= \sum_{k=1}^{n} a_{ik}b_{kj}$$

Also have the i-th row of AB is equal to the i-th row of A times B

Examples

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \end{bmatrix}$$

$$\underset{2\times 1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} \underset{1\times 2}{\begin{bmatrix} 1 & 2 \end{bmatrix}} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

$$\underset{2\times 2}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$$

$\therefore$ matrix mult does not obey the commutative law

i.e. $AB \neq BA$ in general

$(AB)C = A(BC)$ when both sides are defined

Proof: Have to show

$(AB) \times$ j-th col of C $=$ $A \times$ j-th col of BC

$= A \times (B \times$ j-th col of $C)$

This reduces to proving the result for a column matrix C

i.e. $(AB)X = A(BX)$ X col. matrix

$\therefore$ Associative Law: $(AB)C = A(BC)$ holds for matrix multiplication

A matrix $A = [a_{ij}]$ is said to be square if it has the same number of rows and columns

i.e. $A \in \mathbb{R}^{n \times n}$

$$A^2 = AA, \quad A^3 = A \cdot AA = A^2 A, \ldots, A^{n+1} = A^n A$$

$$A^0 = I \quad \text{identity matrix} \quad \text{(by def)}$$

$$= \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \end{bmatrix} = I \qquad IA = A$$

also have

$$AI = A$$

$$A^k A^l = A^{k+l} = A^l A^k$$

$$Ax = y$$ system of equations in matrix form

if $BA = I$, then $x = By$

if also $AB = I$, then $x = By$

$$\Rightarrow Ax = A(By) = (AB)y = Iy = y$$

$\therefore$ if there is a matrix B such that

$$AB = I_m \text{ and } BA = I_n$$

$I_m =$ Identity $m \times m$ matrix.

then $Ax = y$ has a unique solution, namely $x = By$

The matrix B is called the inverse of A and is denoted by A^{-1}.

$$B = A^{-1} \Leftrightarrow AB = I \text{ and } BA = I$$

$(A^{-1})^{-1} = A$ and $(AB)^{-1} = B^{-1}A^{-1}$ if A, B invertible

Indeed, $B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$ and
$ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\Delta = ad - bc \neq 0$ then

A^{-1} exists and $A^{-1} = \begin{bmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Transpose of a Matrix

The transpose of an $m \times n$ matrix $A = [a_{ij}]$ is the $n \times m$ matrix A^T whose (i,j)-th entry is a_{ji}.

Examples. $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^T = [x_1, \dots, x_n]$, $[x_1, \dots, x_n]^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$(A^T)^T = A$, $(A+B)^T = A^T + B^T$, $(cA)^T = cA^T$

$(AB)^T = B^T A^T$ Hint: Prove it column by column to reduce it to the case A is a row matrix

Decision procedure for deciding the invertibility of a matrix and finding the inverse when possible

Example. $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

Consider $Ax = y$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Try to solve for x using GJ elimination

$$x_1 + 2x_2 = y_1$$
$$2x_1 + 3x_2 = y_2$$

$E_2 - 2E_1$
$$x_1 + 2x_2 = y_1$$
$$-x_2 = y_2 - 2y_1$$

$-E_2$
$$x_1 + 2x_2 = y_1$$
$$x_2 = 2y_1 - y_2$$

$E_1 - 2E_2$
$$x_1 = y_1 - 2(2y_1 - y_2) = -3y_1 + 2y_2$$
$$x_2 = 2y_1 - y_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Claim $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ invertible & $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$

$$Ax = y \Leftrightarrow Ax = Iy$$

Write this system in the form
$$[A \mid I]$$
and perform the row reductions

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad I \qquad\quad A^{-1}$$

A square matrix is invertible $\Longleftrightarrow$ it is row equivalent to the identity matrix