Matrices

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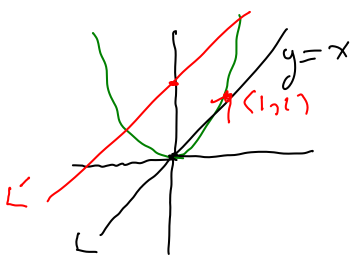
$\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R} \}$

$u \in \mathbb{R}^n$

Define $T_u : \mathbb{R}^n \to \mathbb{R}^n$ by $T_u(x) = u + x$.

T_u is translation by u. $\quad u = T_u(x) - x$ for any x



$L = \{ (x,y) \mid y = x \}$

$= \{ (x,y) \mid x - y = 0 \}$

L has parametric equation

$x = t, \; y = t$

$L = \text{span}((1,-1)) = \{ t(1,-1) \mid t \in \mathbb{R} \}$

$T_{(0,1)}(L) = \{ (0,1) + t(1,-1) \mid t \in \mathbb{R} \} = L'$

Example of a subset of $\mathbb{R}^2$ which is not a subspace

(1) $y = x + 5$ — not a subspace since $(0,0)$ not in the set

(2) $y = x^2$ — parabola P

$P = \{ (x, x^2) \mid x \in \mathbb{R} \}$ not a subspace

since $(1,1)$ on the parabola

but $2(1,1) = (2,2)$ not on the parabola

since $2 \neq 2^2$

u, v vectors in $\mathbb{R}^n$

$u-v$, $u+v$ lin. indep?

$x(u-v)+y(u+v) = 0$ dep. rel

$(x+y)u + (-x+y)v = 0$

suppose u, v are linearly independent

Then $x+y=0$ and $-x+y=0 \Rightarrow x=y=0$

$\Rightarrow$ $u-v$, $u+v$ lin independent

$u-v$, $u+v$, $u+3v$ consider these vectors

same question

$x(u-v)+y(u+v)+z(u+3v) = 0$

$(x+y+z)u + (-x+y+3z)v = 0$

want $x+y+z=0$ and not all $x,y,z=0$

$-x+y+3z=0$ to show lin dep

GJE $\Rightarrow$

$$\begin{array}{lll} x+y+z=0 & x+y+z=0 & x-z=0 \\ 2y+4z=0 & y+2z=0 & y+2z=0 \end{array}$$

$\Rightarrow$ sol. set $= \{ t(1,-2,1) \mid t \in \mathbb{R} \}$

$(1,-2,1)$ non-trivial sol. $\Rightarrow$

$1(u-v) - 2(u+v) + 1(u+3v) = 0$

which is a non-trivial dependence relation amongst the vectors $u-v$, $u+v$, $u+3v$.

$\therefore$ they are linearly dependent.

Matrices

$$a x_1 + b x_2 = y_1$$
$$c x_1 + d x_2 = y_2$$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ given coeff. matrix

Rewrite this equation as

$$\begin{bmatrix} a x_1 + b x_2 \\ c x_1 + d x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Rewrite left-hand side as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

def. of multiplication a 2×2 matrix by a 2×1 matrix

If we let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ then the original system is written as

$$Ax = y$$

Example: $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + 2\cdot 3 \\ 2\cdot 2 + 1\cdot 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$

Now define multiplication of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by a 2×2 matrix $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ as follows:

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \left[A\begin{bmatrix} e \\ g \end{bmatrix}, A\begin{bmatrix} f \\ h \end{bmatrix} \right]$$

$$= \begin{bmatrix} ae+bg & ce+dg \\ af+bh & cf+dh \end{bmatrix}$$

Example 1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$

Example 2. If $\begin{matrix} x_1 + x_2 = y_1 \\ x_1 - x_2 = y_2 \end{matrix}$ then $\begin{matrix} x_1 = \frac{1}{2}y_1 + \frac{1}{2}y_2 \\ x_2 = \frac{1}{2}y_1 - \frac{1}{2}y_2 \end{matrix}$

which shows that if $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

then $Ax = y \Rightarrow x = By$. To show the converse

namely that $x = By \Rightarrow Ax = y$ this

amounts to checking that $x_1 = \frac{1}{2}y_1 + \frac{1}{2}y_2$, $x_2 = \frac{1}{2}y_1 - \frac{1}{2}y_2$ is

a solution of $x_1 + x_2 = y_1$, $x_1 - x_2 = y_2$:

$$x_1 + x_2 = \left(\tfrac{1}{2}y_1 + \tfrac{1}{2}y_2\right) + \left(\tfrac{1}{2}y_1 - \tfrac{1}{2}y_2\right) = y_1$$

$$x_1 - x_2 = \left(\tfrac{1}{2}y_1 + \tfrac{1}{2}y_2\right) - \left(\tfrac{1}{2}y_1 - \tfrac{1}{2}y_2\right) = y_2$$

Thus $\begin{matrix} x_1 = \frac{1}{2}y_1 + \frac{1}{2}y_2 \\ x_2 = \frac{1}{2}y_1 - \frac{1}{2}y_2 \end{matrix}$ is the unique solution of $\begin{matrix} x_1 + x_2 = y_1 \\ x_1 - x_2 = y_2 \end{matrix}$

More generally, let $Ax = y$, $x = By$ with

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

We claim that $A(By) = (AB)y$

Indeed,

$$A(By) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} ey_1+fy_2 \\ gy_1+hy_2 \end{bmatrix}$$

$$= \begin{bmatrix} a(ey_1+fy_2)+b(gy_1+hy_2) \\ c(ey_1+fy_2)+d(gy_1+hy_2) \end{bmatrix}$$

$$= \begin{bmatrix} (ae+bg)y_1+(af+bh)y_2 \\ (ce+dg)y_1+(cf+dh)y_2 \end{bmatrix}$$

$$= \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = (AB)y$$

Let's use this to prove the following result:

Theorem. If $\Delta = ad-bc \neq 0$ then the system

$$ax_1+bx_2 = y_1$$
$$cx_1+dx_2 = y_2$$

has the unique solution

$$x_1 = \frac{d}{\Delta}y_1 - \frac{b}{\Delta}y_2$$
$$x_2 = -\frac{c}{\Delta}y_1 + \frac{a}{\Delta}y_2$$

Proof. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

We have $AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ (the identity 2×2 matrix)
and $Ix = x$, $Iy = y$ (I acts like the number 1)

First let us show that $x = By$ is a solution of the system $Ax = y$. This follows from

$$Ax = A(By) = (AB)y = Iy = y$$

Now let us show that this is the only solution. If $Ax = Az = y$ then

$$B(Ax) = B(Az)$$

so that $$(BA)x = (BA)z$$

But $BA = I$ which gives $Ix = Iz$ and hence that $x = z$. Thus

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x = By = \begin{bmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} (dy_1 - by_2)/\Delta \\ (-cy_1 + ay_2)/\Delta \end{bmatrix}$$

as the unique solution of $Ax = y$.

<u>Remark</u>. The matrix B is called the inverse of A and is denoted by A^{-1}. Thus, when A^{-1} exists, the solution of $Ax = y$ is $x = A^{-1}y$ just like what happens with numbers.