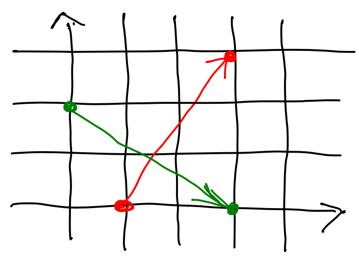
In this course we will study 2 kinds of vectors: geometrical and numerical. Roughly speaking, vectors are carriers of information that can be superposed (added) and scaled (multiplied by a scalar (= number)).

Before giving formal definitions, let's play a game that will give an introduction to vectors. The game is called the racetrack game and is played on a grid formed by 2 sets of parallel lines. Players move from one grid point to another according to fixed rules.

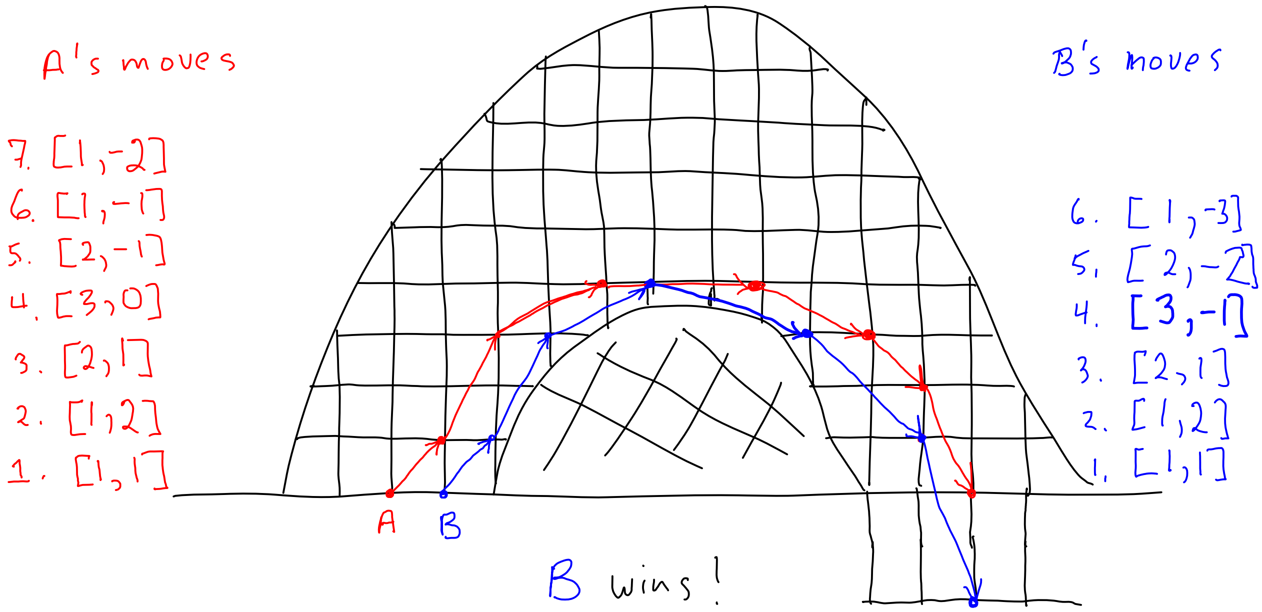
A move is an instruction $[a, b]$: "Go a units horizontally and b units vertically. The signs of a, b indicate the direction (right, up = + left, down = −).



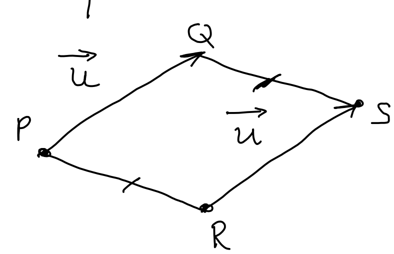
The game is played in a fixed region and there is a starting line and a finishing line. The rules are as follows

1. If $[a,b], [c,d]$ are successive moves by a player then $|a-c| \le 1, |b-d| \le 1$.
2. Two player cannot occupy the same grid point at the same time.
3. The line segment joining two successive positions has to lie entirely in the given region

The first player to cross the finishing line wins. For players crossing the finishin line on the same turn, the player going the farthest past the line wins. Let's play a game with 2 players same turn, the player going the farthest past the line wins. Let's play a game with 2 players A and B



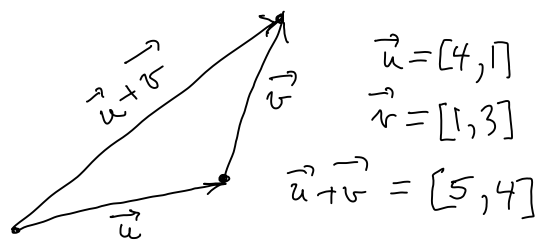
In this game there are 3 types of vectors. The ordered pair [a,b] is one type, namely, a numerical vector. It is used to represent another type of vector, namely, a displacement or translation $\vec{u}$ = "Go a units horizontally and b units vertically". If we perform this displacement starting at a point P we arrive at a point Q. The line segment joining P and Q with an arrow tip at Q to denote the direction from P to Q. This directed line segment is denoted by $\overrightarrow{PQ}$.



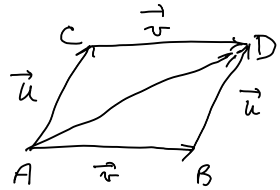
If we perform $\vec{u}$ starting at a point R we arrive at a point S. Directed line segments which are obtained from the same displacement this way are said to be equivalent. Two directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are equivalent if and only if the line segments PQ and RS are equal in length and are parallel. Every directed line segment $\overrightarrow{PQ}$ defines a unique displacement $\vec{u}$.

A displacement is refered to as a free geometrical vector since you are free to apply it at any point, while a directed line segment is refered to as a bound geometrical vector. It can be thought of as a free geometrical vector attached to a point. The moves in the racetrack game are examples of bound vectors; a free vector $\vec{u} = [a, b]$ applied to a point.

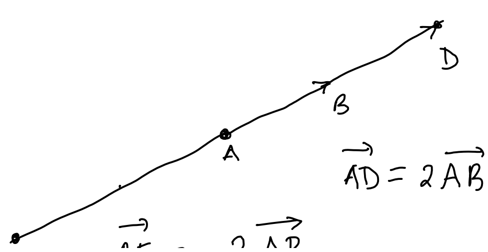
If we perform successive displacements $\vec{u} = [a, b]$, $\vec{v} = [c, d]$ we obtain the displacement $[a+c, b+d]$ which we denote by $\vec{u} + \vec{v}$. This rule of addition is known as the triangle law. See the diagram to see why



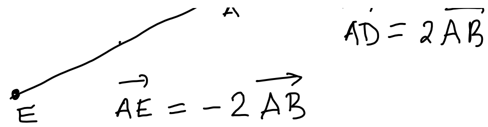
Two bound vectors can be added only if they have the same initial point. The sum $\overrightarrow{AB} + \overrightarrow{AC}$ is the vector $\overrightarrow{AD}$ where $\overrightarrow{CD}$ is equivalent to $\overrightarrow{AB}$. This is known as the parallelogram law. This also shows that the addition of geometrical vectors is commutative, that is, $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.



Finally, vectors can be multiplied by scalars. If $\vec{u} = [a, b]$, then $c\vec{u} = [ca, cb]$ where c is a scalar. The bound vector $c\overrightarrow{AB}$ is the vector AD where D lies on the line through A and B, the length of AD $= |c|$ times the length of AB and D lies



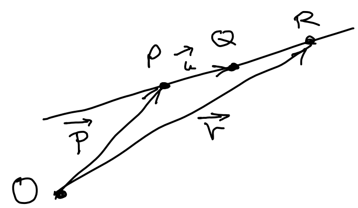
the length of AD $= |c|$ times the length of AB and D lies on the same side of A as B if and only if $c \geq 0$. If the free vector $\vec{u}$ is represented by $\overrightarrow{AB}$, then $c\vec{u}$ is represented by $c\overrightarrow{AB}$.



Bound vectors are used to describe the position of a point with respect to another point. For example, if we choose a point O of our grid as an origin and apply the free vector $\vec{u} = [a, b]$ at O, we get a point P with coordinates (a, b). So (a, b) tells you how to get from O to P. If $Q = (c, d)$ is another point and $\alpha = c - a$, $\beta = d - b$ then the free vector $[\alpha, \beta]$ tells you how to get from P to Q. If $R(x, y)$ is any point on the line through P and Q then $\overrightarrow{PR} = t\,\overrightarrow{PQ}$ for some scalar t.

If $\vec{p}, \vec{u}, \vec{r}$ are the free vectors represented by $\overrightarrow{OP}, \overrightarrow{PQ}, \overrightarrow{OR}$ respectively, we have

$$\vec{r} = \vec{p} + t\vec{u}$$



which is the vector equation of the line through P and Q. Since $\vec{r} = [x, y]$, $\vec{p} = [a, b]$, $\vec{u} = [\alpha, \beta]$, we have

$$[x, y] = [a, b] + t[\alpha, \beta]$$
$$= [a + t\alpha, b + t\beta]$$

or, equivalently, $x = a + t\alpha$, $y = b + t\beta$. The are known as the parametric equations of the line through P and Q. The numerical vector $[\alpha, \beta]$ gives the direction of the line and (a, b) is a point on the line. The midpoint of the line segment PQ would correspond to $t = \frac{1}{2}$. In this case

$$\vec{r} = [a, b] + \tfrac{1}{2}[c - a, d - b] = \left[\frac{a + c}{2}, \frac{b + d}{2}\right]$$

$$= [a,b] + \frac{1}{2}[c-a, a-b] = \left[\frac{\quad}{2}, \frac{\quad}{2}\right]$$

and the coordinates of the midpoint of the line segment joining $P(a,b)$ and $Q(c,d)$ are $\left(\frac{a+b}{2}, \frac{b+d}{2}\right)$.