Final Examination | December 12, 2000 | 189-133A

PART I **Version 2**

1. Let $A = \begin{pmatrix} 1 & 3 & -1 & 2 & -5 \\ 2 & 6 & -2 & 4 & -10 \\ 3 & 9 & -2 & 5 & 7 \\ 4 & 12 & -4 & 11 & -8 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2 \\ 4 \\ 6 \\ 11 \end{pmatrix}$.

   (a) (5%) Find all solutions to $A\vec{x} = \vec{b}$.

   (b) (5%) Find a matrix B so that BA is the row-reduced echelon form of A.

   (c) (3%) Find a basis for the row space of A.

   (d) (3%) Find a basis for the column space of A.

   (e) (3%) Find a basis for the null space of A.

   (f) (3%) Decide whether $\begin{pmatrix} 4 \\ 8 \\ 10 \\ 0 \end{pmatrix}$ is in the column space of A. If it is, express it as a linear combination of the vectors in your basis from part (d). If not, explain why not.

   (g) (3%) Decide whether $\begin{pmatrix} 3 & 9 & -2 & 8 & 19 \end{pmatrix}$ is in the row space of A. If it is, express it as a linear combination of the vectors in your basis from part (c). If not, explain why not.

2. Consider the points $P = (2, 3, 4)$, $Q = (1, -1, 5)$, $R = (6, 4, 4)$ and $S = (2, 5, 7)$ in $\mathbb{R}^3$.

   (a) (3%) Find an equation for the plane containing the points P, Q and R.

   (b) (3%) Say PQ and PR are two sides of a parallelogram and T is its fourth vertex. What are the coordinates of T? What is the area of the parallelogram?

   (c) (3%) What is the volume of the parallelepiped with PQ, PR and PS as three of its edges?

   (d) (3%) What is the shortest distance from the line ℓ_1 through P and Q and the line ℓ_2 through R and S?

   (e) (4%) Find the distance from S to the plane containing P, Q and R.

3. (10%) Let $\mathbf{P}$ be the vector space of polynomials with real coefficients. For each of the following subsets of $\mathbf{P}$, decide whether or not it is a subspace of $\mathbf{P}$. Justify your answers.

   (a) $\{f(x) \in \mathbf{P} | f(4) = 0\}$.

   (b) $\{f(x) \in \mathbf{P} | f(x)$ has degree exactly $3\}$, together with the zero polynomial.

   (c) $\mathbf{P}_3 = \{f(x) \in \mathbf{P} | f$ has degree at most $3\}$, including the zero polynomial.

PART II (Multiple Choice) **Version 2**

Each of the following questions is worth 5% There is only one correct answer in each case.

1. Suppose that A is a 2×2 matrix which has trace -1 and determinant -6. Its eigenvalues are

   (a) -1 and -6, (b) 2 and -3, (c) -2 and 3, (d) 1 and 6.

2. The rank of $\begin{pmatrix} 3 & 2 & 1 & 4 \\ 7 & 0 & 9 & 0 \\ 2 & 5 & 6 & 0 \\ 12 & 7 & 16 & 4 \end{pmatrix}$ is

   (a) 1, (b) 2, (c) 3, (d) 4.

3. Suppose that 2 and -5 are eigenvalues of the matrix A. Which of the following statements is true?

   (a) $2I - A$ is not invertible, and there is a nontrivial solution to $A\vec{v} = -5\vec{v}$.

   (b) $2I - A$ is invertible, and there is a nontrivial solution to $A\vec{v} = -5\vec{v}$.

   (c) $\det(2I + A) = \det(5I - A) = 0$.

   (d) $\det(2I + A) = 0$, and there is a nontrivial solution to $(5I - A)\vec{v} = \vec{0}$.

4. Suppose that $\vec{v}$ and $\vec{w}$ are in $\mathbb{R}^3$ and $\{\vec{v}, \vec{w}\}$ is independent. Which of the following sets is independent?

   (a) $\{\vec{0}, \vec{v}, \vec{w}\}$.

   (b) $\{\vec{v}, \vec{w}, 3\vec{v} - 4\vec{w}\}$.

   (c) $\{\vec{v} - \vec{w}, \vec{v} + \vec{w}, 2\vec{v}\}$.

   (d) $\{\vec{v}, \vec{w}, \vec{v} \times \vec{w}\}$.

5. If $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{pmatrix}$, then the adjoint $adj(A)$ is:

   (a) $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 2 & 4 & 2 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 0 & -1 \\ -6 & 0 & 3 \\ 10 & 0 & -5 \end{pmatrix}$, (c) $\begin{pmatrix} 2 & -6 & 10 \\ 0 & 0 & 0 \\ -1 & 3 & -5 \end{pmatrix}$, (d) $\begin{pmatrix} 2 & 6 & 10 \\ 0 & 0 & 0 \\ -1 & -3 & -5 \end{pmatrix}$.

6. The graph of $4x^2 - 24xy + 11y^2 = 1$ is either an ellipse or a hyperbola. Decide which. Also, a new orthonormal basis $\{\vec{v}, \vec{w}\}$ for $\mathbb{R}^2$ is given in each of the possible answers so that with respect to this basis the curve is in standard form. Choose the answer which has the correct shape and new basis.

   (a) Ellipse; $\vec{v} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$, $\vec{w} = \begin{pmatrix} \frac{4}{5} \\ \frac{-3}{5} \end{pmatrix}$.

   (b) Hyperbola; $\vec{v} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$, $\vec{w} = \begin{pmatrix} \frac{4}{5} \\ \frac{-3}{5} \end{pmatrix}$.

   (c) Ellipse; $\vec{v} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix}$, $\vec{w} = \begin{pmatrix} \frac{-3}{5} \\ \frac{4}{5} \end{pmatrix}$.

   (d) Hyperbola; $\vec{v} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix}$, $\vec{w} = \begin{pmatrix} \frac{-3}{5} \\ \frac{4}{5} \end{pmatrix}$.

7. Suppose that A and B are 4×4 matrices and that $\det(A) = \det(B) = -\frac{1}{2}$. Then $\det(-A^3BA^T(-3B^2)(-A)^{-1})$ is

   (a) $-\dfrac{81}{64}$, (b) $\dfrac{81}{64}$, (c) $\dfrac{3}{64}$, (d) $-\dfrac{3}{64}$.

8. Which of the points mentioned is on the same line as $P = (1, 0, 5)$ and $Q = (3, 1, -2)$?

   (a) $(0, -3, 17)$, (b) $(9, 4, -23)$, (c) $(-9, -4, 23)$, (d) $(0, 3, -17)$.

9. Consider the subspace of $\mathbb{R}^3$ spanned by $\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Which of the following vectors is orthogonal to this subspace?

   (a) $\begin{pmatrix} 3 \\ 8 \\ 3 \end{pmatrix}$, (b) $\begin{pmatrix} 3 \\ 8 \\ -3 \end{pmatrix}$, (c) $\begin{pmatrix} 3 \\ -8 \\ 3 \end{pmatrix}$, (d) $\begin{pmatrix} -3 \\ 8 \\ 3 \end{pmatrix}$.

10. When does the system $\begin{array}{rcrcrcl} x_1 & + & 2ax_2 & & & = & 0 \\ 2x_1 & + & 6x_2 & + & ax_3 & = & 0 \\ & & -8x_2 & + & 4ax_3 & = & 0 \end{array}$ have a nontrivial solution?

    (a) Only when $a = 0$ or $a = 2$.

    (b) For any real number a.

    (c) Only if $a = 0$.

    (d) Only when $a = 0$ or $a = -2$.

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-133A

VECTORS, MATRICES & GEOMETRY

Examiner: Professor J. Loveys
Associate Examiner: Professor O. Kharlampovich

Date: Tuesday, December 12, 2000
Time: 9:00 A.M. - 12:00 Noon

INSTRUCTIONS

Calculators are not permitted.

This examination is in TWO parts. Part I is to be answered on the booklet(s) provided.

Part II is to be answered on the multiple choice answer sheet (scantron).

This examination is **Version 2**. Make sure that your name, student number and version number are clearly indicated on the examination paper, any booklet(s) used and the multiple choice answer sheet.

This exam comprises the cover and three pages of questions.