1. (a) Find an equation for the plane containing the line $(x, y, z) = (1 + t, 2 + 2t, 3 - t)$ and parallel to the line $(x, y, z) = (s, s, s)$.

(b) Find the distance between the two lines in 1(a).

2. (a) Find, in vector parametric form, the equation of the line L of intersection of the two planes

$$x + 2y + 3z = 1 \quad \text{and} \quad 2x + 5y - z = 2.$$

(b) Find the equation of the plane containing the line L in 2(a) and perpendicular to the line

$$x = 1 + 3t, y = 1 + 7t, z = 1 + 2t.$$

3. (a) Find all values of k for which the matrix $\begin{bmatrix} k & -2 & 3 \\ k-4 & k-4 & -5 \\ k & -2 & k+5 \end{bmatrix}$ is invertible.

(b) For which values of k is the system

$$\begin{aligned} kx - 2y + 3z &= 1 \\ (k-4)x + (k-4)y - 5z &= 1 \\ kx - 2y + (k+5)z &= 1 \end{aligned}$$

solvable?

4. Let $A = \begin{bmatrix} 0 & 1 \\ -3 & 9 \end{bmatrix}$

(a) Write A^{-1} as a product of elementary matrices.

(b) Write A as a product of elementary matrices.

5. Let $A = \begin{bmatrix} 1 & 2 & 2 & 1 & 1 \\ 2 & 4 & 5 & 3 & 2 \\ 3 & 6 & 7 & 5 & 3 \\ 4 & 8 & 9 & 5 & 4 \end{bmatrix}$.

(a) Bring A to row reduced echelon form.

(b) Find bases for (i) the row space, (ii) the column space and (iii) the null space of A.

6. Let u_1, u_2, u_3 be vectors in $\mathbb{R}^n$ and let $v_1 = u_1, v_2 = u_1 + 2u_2, v_3 = u_1 + 2u_2 - u_3$.

(a) If u_1, u_2, u_3 are linearly independent, show that v_1, v_2, v_3 are linearly independent.

(b) If v_1, v_2, v_3 are linearly independent, show that u_1, u_2, u_3 are linearly independent.

7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with

$$T\left(\begin{bmatrix}-4\\-3\end{bmatrix}\right) = \begin{bmatrix}3\\5\end{bmatrix}, \quad T\left(\begin{bmatrix}-3\\-2\end{bmatrix}\right) = \begin{bmatrix}4\\-5\end{bmatrix}.$$

(a) Find the matrix of T with respect to the standard basis.

(b) Show that T^{-1} exists and find its matrix with respect to the standard basis.

8. (a) For which values of c is the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & c \\ 0 & 0 & 2 \end{bmatrix}$ diagonalizable.

(b) If $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$,

(i) find the characteristic polynomial of A and the eigenvalues of A; (Hint: -2 is an eigenvalue)

(continued on next page)

(ii) find a basis of each eigenspace;

(iii) determine whether or not there is an matrix P such that $P^{-1}AP$ is a diagonal matrix. Exhibit P if it exists but you don't have to compute P^{-1} explicitly.

9. Let $A = \begin{bmatrix} 3 & 2 \\ 5 & 6 \end{bmatrix}$

(a) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

(b) Find a matrix B with $B^3 = A$.

10. (a) If A is a diagonalizable 2×2 matrix with $A^3 = -I$, where I is the identity matrix, show that $A = -I$.

(b) Using the fact that $A^2 - \operatorname{tr}(A)A + \det(A)I = 0$ for any 2×2 matrix A, find a 2×2 matrix A with $A^2 - A + I = 0$. Show that $A^3 = -I$.

11. Let $W = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}\right)$

(a) Find an orthognormal basis for W.

(b) Find an basis for $W^{\perp}$ and use this to find a homogeneous system of equations whose solution space is W.

12. Let $A = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$.

(a) Find an orthogonal matrix P such that $P^T AP$ is a diagonal matrix.

(b) Using (a), show that $5x^2 - 6xy + 5y^2 = 2$ is the equation of an ellipse. Find the length and a direction vector of each axis.

NAME:

STUDENT NUMBER:

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 133

Vectors Matrices and Geometry

Examiner: Professor J. Labute
Associate Examiner: Professor I. Klemes

Date: Wednesday, December 10, 2003
Time: 9:00 A.M. - 12:00 P.M.

INSTRUCTIONS

Attempt all questions.
All questions are of equal value.
Answer all questions on the pages provided.
Show and justify all your work.
Calculators, books and notes are not permitted.
All matrices are real matrices.

This exam comprises the cover and 17 pages with 12 questions and 4 additional blank pages.

1	2	3	4	5	6	7	8	9	10	11	12