

Computing discrete convolutions with verified accuracy via Banach algebras and the FFT

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Abstract

In this paper, we introduce a method to compute rigorous component-wise enclosures of discrete convolutions using the fast Fourier transform, the properties of Banach algebras and interval arithmetic. The purpose of this new approach is to improve the implementation and the applicability of computer-assisted proofs performed in weighed ℓ^1 Banach algebras of Fourier/Chebyshev sequences, whose norms are known to be numerically unstable. We introduce some application examples, in particular a rigorous a posteriori error analysis for a steady state in the quintic Swift-Hohenberg PDE.

Keywords

Discrete Convolutions · Banach Algebras · Fast Fourier Transform
Interval Arithmetic · Rigorously Verified Numerics · Quintic Swift-Hohenberg PDE

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1 Introduction

In this paper, we introduce a method to compute discrete convolutions with verified accuracy using the fast Fourier transform (FFT), the properties of Banach algebras of bi-infinite complex valued sequences and interval arithmetic. Our motivation comes from the field of rigorously verified numerics in dynamics (e.g. see [8, 16, 19, 21, 22, 25, 27, 30]), which aims at obtaining computer-assisted proofs (CAPs) of existence of solutions of differential equations (ODE, PDEs, delay equations, etc) and discrete dynamical systems (iterations of finite and infinite dimensional maps). One of the common approach in obtaining the CAPs for differential equations is to represent solutions using Fourier/Chebyshev series, and to apply the contraction mapping theorem on a ball centered at a numerical approximation in a Banach space of Fourier/Chebyshev coefficients with weighed ℓ^1 norms (e.g. see [9, 15, 26, 28]). More explicitly, the Banach spaces considered in the aforementioned papers are sequence spaces of the form

$$\ell_\omega^1 \stackrel{\text{def}}{=} \{a = \{a_k\}_{k \in \mathbb{Z}} : a_k \in \mathbb{C} \text{ and } \|a\|_\omega < \infty\}, \quad (1)$$

with norm

$$\|a\|_\omega \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} |a_k| \omega_k, \quad (2)$$

where $\omega = \{\omega_k\}_{k \in \mathbb{Z}}$ is a sequence of positive weights satisfying the sub-multiplicative property $\omega_{n+k} \leq \omega_n \omega_k$ for all $n, k \in \mathbb{Z}$. The advantage of using the sequence space (1) in the CAPs is essentially twofold. First, ℓ_ω^1 has a known dual (a weighed ℓ^∞ space) which facilitates estimating norms of functionals and operators. Second, the sub-multiplicative property of the weights ensures that ℓ_ω^1 is a Banach algebra under discrete convolutions (see Section 3), which facilitates performing the nonlinear analysis. However, choosing the Banach space ℓ_ω^1 in performing the CAPs has the disadvantage that its norm $\|\cdot\|_\omega$ can be unstable with respect to a given sequence of weights. For instance, in the papers [12, 17, 26] the numerical instability of the ℓ^1 norms with geometrically growing weights prevented obtaining some CAPs (e.g. see the discussion in Section 6 of [12] or the discussion in Section 6.3 in [26]). The goal of this paper is precisely to address this issue. In order to formalize the problem, we need a bit of notation and background on discrete convolutions.

Assume that two functions $u_1, u_2 : \mathbb{R} \rightarrow \mathbb{R}$ with period $\frac{2\pi}{L}$ (that is with frequency $L > 0$) have absolutely converging Fourier series expansions

$$u_1(t) = \sum_{k \in \mathbb{Z}} a_k^{(1)} e^{ikLt} \quad \text{and} \quad u_2(t) = \sum_{k \in \mathbb{Z}} a_k^{(2)} e^{ikLt}.$$

Then, their product $u_1(t)u_2(t)$ also has an absolutely converging Fourier series expansion, which is given by

$$u_1(t)u_2(t) = \sum_{k \in \mathbb{Z}} (a^{(1)} * a^{(2)})_k e^{ikLt},$$

where, given two sequences $a = \{a_k\}_{k \in \mathbb{Z}}$ and $b = \{b_k\}_{k \in \mathbb{Z}}$, $a * b$ denotes their *discrete convolution* defined component-wise by

$$(a * b)_k = \sum_{\substack{k_1 + k_2 = k \\ k_1, k_2 \in \mathbb{Z}}} a_{k_1} b_{k_2}. \quad (3)$$

Remark 1.1. *The approach presented in the present paper can also help controlling the coefficients of product of functions represented by Chebyshev series. Indeed, as Chebyshev series are Fourier series “in disguise” [1], their product naturally leads to discrete convolutions (e.g. see [18]).*

For many applications and for some computer-assisted proofs, we are interested in computing product of trigonometric polynomials with high but finite degree, that is Fourier series which have only finitely many non zero coefficients. More explicitly, we consider $\frac{2\pi}{L}$ -periodic trigonometric polynomials $u_i(t)$ ($i = 1, \dots, p$) of degree $M - 1$ defined by

$$u_i(t) = \sum_{|k| < M} a_k^{(i)} e^{ikLt}, \quad i = 1, \dots, p. \quad (4)$$

As the following result demonstrates, the product of p -periodic trigonometric polynomials of degree $M - 1$ has degree $p(M - 1)$.

Lemma 1.2. *For $i = 1, \dots, p$, let u_i the $\frac{2\pi}{L}$ -periodic trigonometric polynomials of degree $M - 1$ with expansion (4). The Fourier expansion of the product $u_1 u_2 \cdots u_p$ satisfies*

$$u_1(t) u_2(t) \cdots u_p(t) = \sum_{|k| \leq p(M-1)} (a^{(1)} * a^{(2)} * \cdots * a^{(p)})_k e^{ikLt}. \quad (5)$$

In other words, the function $u_1 u_2 \cdots u_p$ is a trigonometric polynomial of degree $p(M - 1)$.

Proof. Let $k \in \mathbb{Z}$ such that $k = k_1 + \cdots + k_p$ for some $k_1, \dots, k_p \in \mathbb{Z}$ with $|k_i| < M$ ($i = 1, \dots, p$). Hence,

$$k = k_1 + \cdots + k_p \in \{-p(M - 1), \dots, p(M - 1)\},$$

and therefore $|k| \leq p(M - 1)$. We conclude that

$$\begin{aligned} u_1(t) u_2(t) \cdots u_p(t) &= \sum_{k \in \mathbb{Z}} (a^{(1)} * a^{(2)} * \cdots * a^{(p)})_k e^{ikLt} \\ &= \sum_{|k| \leq p(M-1)} (a^{(1)} * a^{(2)} * \cdots * a^{(p)})_k e^{ikLt}. \end{aligned} \quad \square$$

We are ready to state the goal of the present paper.

Statement of the problem: *Given p finite sequences of Fourier/Chebyshev coefficients $\{a_k^{(i)}\}_{|k| < M}$ ($i = 1, \dots, p$) combine interval arithmetic, the fast Fourier transform (FFT) and Banach algebras to obtain rigorous component-wise enclosures for the discrete convolution*

$$(a^{(1)} * a^{(2)} * \cdots * a^{(p)})_k = \sum_{\substack{k_1 + k_2 + \cdots + k_p = k \\ |k_i| < M}} a_{k_1}^{(1)} a_{k_2}^{(2)} \cdots a_{k_p}^{(p)}, \quad \text{for } |k| \leq p(M - 1). \quad (6)$$

The statement of the problem comes from the rather common need in rigorous numerics to compute interval enclosures for terms of the form (6) (e.g. see [4, 5, 6, 10, 31, 32]). While

the FFT algorithm [29] is a fantastic tool to evaluate quickly discrete convolutions (e.g. see Section 2), it often fails to recover the geometric/algebraic decay of the tail (i.e. the terms in (6) corresponding to $M \leq |k| \leq p(M-1)$) of a convolution (e.g. see the example in Section 5.1) due to round-off errors. This property implies that the weighted ℓ^1 norms of the discrete convolutions may blow-up for geometrically growing weights (e.g. when $\omega_k = \nu^{|k|}$ for some $\nu > 1$) (see Tables 4 and 5). This is a major hurdle in obtaining the CAPs in the category of analytic functions, as sometimes taking ν rather large is necessary to obtain a contraction mapping. In order to fix this issue in the tail, we use the fundamental property of a given Banach algebra $(X, *)$ (i.e. that $\|x * y\|_X \leq \|x\|_X \|y\|_X$, for all $x, y \in X$) to obtain the proper decay in the tail. This has the effect of stabilizing the sensitivity of the ℓ^1 norms in obtaining the rigorous bounds for the computer-assisted proofs.

Before proceeding any further, we urge to mention that the present work is by no means the first time that the FFT algorithm and interval arithmetic are combined to rigorously compute discrete convolutions (e.g. see [3, 11, 14, 8]). However, we believe that our new proposed approach of combining the FFT algorithm, interval arithmetic and theoretical Banach algebra estimates to obtain rigorous component-wise enclosure (6) is new, and that it could benefit the rigorous numerics community.

Remark 1.3. *The idea introduced in this paper can be generalized to rigorously enclose components of discrete convolutions of multidimensional sequences in the Banach space*

$$\ell_\omega^1 \stackrel{\text{def}}{=} \{a = \{a_\alpha\}_{\alpha \in \mathbb{Z}^d} : a_\alpha \in \mathbb{C} \text{ and } \|a\|_\omega < \infty\}, \quad (7)$$

where $d \in \mathbb{N}$ is the dimension of the space on which solutions of the differential equations are defined, $\|a\|_\omega \stackrel{\text{def}}{=} \sum_{\alpha \in \mathbb{Z}^d} |a_\alpha| \omega_\alpha$ and $\omega = \{\omega_\alpha\}_{\alpha \in \mathbb{Z}^d}$ is a sequence of positive weights satisfying the sub-multiplicative property $\omega_{\alpha+\beta} \leq \omega_\alpha \omega_\beta$ for all $\alpha, \beta \in \mathbb{Z}^d$. It can be showed that ℓ_ω^1 is a Banach algebra under the discrete convolution defined by

$$(a * b)_\alpha = \sum_{\substack{\beta+\gamma=\alpha \\ \beta, \gamma \in \mathbb{Z}^d}} a_\beta b_\gamma = \sum_{\beta \in \mathbb{Z}^d} a_\beta b_{\alpha-\beta}.$$

The paper is organized as follows. In Section 2, we present the background of how to use the FFT to compute discrete convolutions. In Section 3, we present how to use the Banach algebra property of the Banach space ℓ_ω^1 to obtain an alternative method to compute interval enclosures for terms of the form (6), which recover a similar decay rate in the tail as the one of the inputs. In Section 4, we combine the two approaches of Sections 2 and 3 to refine the rigorous enclosures of the discrete convolutions. Finally in Section 5, we introduce some application examples, where compute rigorously discrete convolutions of degree 3 and 20, and we present a rigorous aposteriori error analysis for a steady state in the quintic Swift-Hohenberg PDE.

2 Convolution enclosures via the FFT and interval arithmetic

In this section, denote by M and p the order of the Fourier series and the power of the nonlinearity, respectively. Following closely the presentation of [11], we introduce the theory to compute discrete convolutions of the form (6) with the discrete Fourier transform (DFT).

Once this is done, we combine the FFT algorithm (an efficient implementation of the DFT) and interval arithmetic to compute rigorous enclosures of discrete convolutions of the form (6).

Definition 2.1. Given $b = (b_0, \dots, b_{2M-2}) \in \mathbb{C}^{2M-1}$, define its *discrete Fourier transform* $\mathcal{F}(b) \in \mathbb{C}^{2M-1}$ by

$$a_k = \mathcal{F}_k(b) \stackrel{\text{def}}{=} \sum_{j=0}^{2M-2} b_j e^{-2\pi i \left(\frac{jk}{2M-1}\right)}, \quad \text{for } k = -M+1, \dots, M-1.$$

Definition 2.2. Given $a = (a_k)_{|k|<M} = (a_{-M+1}, \dots, a_{M-1}) \in \mathbb{C}^{2M-1}$, define its *inverse discrete Fourier transform* $\mathcal{F}^{-1}(a) \in \mathbb{C}^{2M-1}$ by

$$b_j = \mathcal{F}_j^{-1}(a) \stackrel{\text{def}}{=} \sum_{k=-M+1}^{M-1} a_k e^{2\pi i \left(\frac{jk}{2M-1}\right)}, \quad \text{for } j = 0, \dots, 2M-2.$$

Given $a^{(i)} = (a_k^{(i)})_{|k|<M} \in \mathbb{C}^{2M-1}$, we extend it to eliminate the so-called *aliasing error* (see the second term in (11)). Hence, define $\tilde{a}^{(i)} \in \mathbb{C}^{2pM-1}$ by

$$\tilde{a}_j^{(i)} = \begin{cases} a_j^{(i)}, & \text{for } |j| < M \\ 0, & \text{for } M \leq |j| \leq pM-1, \end{cases} \quad (8)$$

that is we *pad* the vector $a^{(i)}$ with $(p-1)M$ zeros before and after. Then define $\tilde{b}^{(i)} = (b_0^{(i)}, \dots, b_{2pM-2}^{(i)}) \in \mathbb{C}^{2pM-1}$ component-wise by

$$\tilde{b}_j^{(i)} \stackrel{\text{def}}{=} \mathcal{F}_j^{-1}(\tilde{a}^{(i)}) = \sum_{k=-pM+1}^{pM-1} \tilde{a}_k^{(i)} e^{2\pi i \left(\frac{jk}{2pM-1}\right)}, \quad \text{for } j = 0, \dots, 2pM-2. \quad (9)$$

Define $\tilde{b}^{(1)} \hat{*} \dots \hat{*} \tilde{b}^{(p)}$ to be the component-wise product of the vectors $\tilde{b}^{(1)}, \dots, \tilde{b}^{(p)}$, that is

$$(\tilde{b}^{(1)} \hat{*} \dots \hat{*} \tilde{b}^{(p)})_j \stackrel{\text{def}}{=} \tilde{b}_j^{(1)} \dots \tilde{b}_j^{(p)}, \quad j = 0, \dots, 2pM-2. \quad (10)$$

Hence, for $k = -pM+1, \dots, pM-1$,

$$\begin{aligned} \mathcal{F}_k(\tilde{b}^{(1)} \hat{*} \dots \hat{*} \tilde{b}^{(p)}) &= \sum_{j=0}^{2pM-2} \tilde{b}_j^{(1)} \dots \tilde{b}_j^{(p)} e^{-2\pi i \left(\frac{jk}{2pM-1}\right)} \\ &= \sum_{j=0}^{2pM-2} \prod_{i=1}^p \left(\sum_{k_i=-pM+1}^{pM-1} \tilde{a}_{k_i}^{(i)} e^{2\pi i \left(\frac{jk_i}{2pM-1}\right)} \right) e^{-2\pi i \left(\frac{jk}{2pM-1}\right)}. \end{aligned}$$

Letting

$$\begin{aligned}
S_k(j) &\stackrel{\text{def}}{=} \prod_{i=1}^p \left(\sum_{k_i=-pM+1}^{pM-1} \tilde{a}_{k_i}^{(i)} e^{2\pi i \left(\frac{jk_i}{2pM-1} \right)} \right) e^{-2\pi i \left(\frac{jk}{2pM-1} \right)} \\
&= \sum_{\substack{k_1+\dots+k_p=k \\ |k_i|<M}} a_{k_1}^{(1)} \dots a_{k_p}^{(p)} + \sum_{\ell=1}^p \left(\sum_{\substack{k_1+\dots+k_p=k \pm \ell(2pM-1) \\ |k_i|<M}} a_{k_1}^{(1)} \dots a_{k_p}^{(p)} \right) \\
&\quad + \sum_{\substack{k_1+\dots+k_p \notin \{k \pm \ell(2pM-1): \ell=0, \dots, p\} \\ |k_i|<M}} a_{k_1}^{(1)} \dots a_{k_p}^{(p)} e^{2\pi i \left(\frac{k_1+\dots+k_p-k}{2pM-1} \right) j},
\end{aligned}$$

we obtain that

$$\begin{aligned}
\mathcal{F}_k(\tilde{b}^{(1)} \hat{*} \dots \hat{*} \tilde{b}^{(p)}) &= \sum_{j=0}^{2pM-2} S_k(j) \\
&= (2pM-1) \sum_{\substack{k_1+\dots+k_p=k \\ |k_i|<M}} a_{k_1}^{(1)} \dots a_{k_p}^{(p)} \\
&\quad + (2pM-1) \sum_{\ell=1}^p \left(\sum_{\substack{k_1+\dots+k_p=k \pm \ell(2pM-1) \\ |k_i|<M}} a_{k_1}^{(1)} \dots a_{k_p}^{(p)} \right) \\
&\quad + \sum_{\substack{k_1+\dots+k_p \notin \{k \pm \ell(2pM-1): \ell=0, \dots, p\} \\ |k_i|<M}} a_{k_1}^{(1)} \dots a_{k_p}^{(p)} \left(\sum_{j=0}^{2pM-2} e^{2\pi i \left(\frac{k_1+\dots+k_p-k}{2pM-1} \right) j} \right). \tag{11}
\end{aligned}$$

Euler's formula gives that for $k_1 + \dots + k_p - k \not\equiv 0 \pmod{2pM-1}$,

$$\sum_{j=0}^{2pM-2} e^{2\pi i \left(\frac{k_1+\dots+k_p-k}{2pM-1} \right) j} = 0.$$

Hence, the third sum in (11) is zero. As far as the second sum in (11) is concerned, observe that $|k_1|, \dots, |k_p| < M$ and that $|k| \leq p(M-1)$, and therefore

$$k_1 + \dots + k_p - k \in \{-2p(M-1), \dots, 2p(M-1)\}.$$

Hence, for the equality $k_1 + \dots + k_p = k \pm \ell(2pM-1)$ to be satisfied for a choice of $\ell \in \{1, \dots, p\}$, one must have that

$$\ell(2pM-1) = k_1 + \dots + k_p - k \in \{-2p(M-1), \dots, 2p(M-1)\},$$

which is impossible because $2pM-1 > 2p(M-1)$. Hence, the second sum of (11) is zero. Therefore, we can conclude that

$$\sum_{\substack{k_1+\dots+k_p=k \\ |k_1|, \dots, |k_p|<M}} a_{k_1}^{(1)} \dots a_{k_p}^{(p)} = \frac{1}{2pM-1} \cdot \mathcal{F}_k(\tilde{b}^{(1)} \hat{*} \dots \hat{*} \tilde{b}^{(p)}), \quad \text{for all } |k| \leq p(M-1). \tag{12}$$

Remark 2.3. In [11], we padded the vectors $a^{(i)}$ as

$$\tilde{a}_j^{(i)} = \begin{cases} a_j^{(i)}, & \text{for } |j| < M \\ 0, & \text{for } M \leq |j| \leq \delta_p M - 1, \end{cases}$$

where

$$\delta_p \stackrel{\text{def}}{=} \begin{cases} \frac{p+1}{2}, & \text{if } p \text{ is odd} \\ \frac{p+2}{2}, & \text{if } p \text{ is even,} \end{cases}$$

because we only considered the cases $|k| < M$ in (6).

The inverse discrete Fourier transforms and the discrete Fourier transforms required in the computations of (9) and (12) can be computed efficiently and rigorously using the FFT algorithm (e.g. see [2]) and interval arithmetic (e.g. [20]) for instance using the function `verifyfft.m` in INTLAB [23].

3 Convolution enclosures via Banach algebras

We begin this section by defining the notion of a Banach algebra.

Definition 3.1. A *Banach algebra* is a Banach space X with a multiplication operation $*$: $X \times X \rightarrow X$ that satisfies

$$\begin{aligned} x * (y * z) &= (x * y) * z \\ (x + y) * z &= x * z + y * z, \quad x * (y + z) = z * y + x * z, \\ \alpha(x * y) &= (\alpha x) * y = x * (\alpha y) \\ \|x * y\| &\leq \|x\| \|y\|, \end{aligned} \tag{13}$$

for all $x, y, z \in X$ and all scalars α . The Banach algebra is *commutative* if $x * y = y * x$, for all $x, y \in X$.

Recall the definition of the Banach space ℓ_ω^1 in (1) endowed with the norm $\|\cdot\|_\omega$ defined in (2), and the definition of the discrete convolution in (3).

The following definition introduces a property about the weight ω which make ℓ_ω^1 a commutative Banach algebra under discrete convolutions.

Definition 3.2. A sequence of positive real numbers $\omega = \{\omega_k\}_{k \in \mathbb{Z}}$ is an *admissible sequence of weights* if $\omega_{n+k} \leq \omega_n \omega_k$, for all $n, k \in \mathbb{Z}$.

Lemma 3.3. Given ω an admissible sequence of weights and denote $*$ the discrete convolution. Then the pair $(\ell_\omega^1, *)$ is a commutative Banach algebra.

Proof. We only demonstrate that $\|a * b\|_\omega \leq \|a\|_\omega \|b\|_\omega$ for all $a, b \in \ell_\omega^1$. Indeed,

$$\begin{aligned}
\|a * b\|_\omega &= \sum_{k \in \mathbb{Z}} |(a * b)_k| \omega_k \\
&= \sum_{k \in \mathbb{Z}} \left| \sum_{k_1 \in \mathbb{Z}} a_{k_1} b_{k-k_1} \right| \frac{\omega_k}{\omega_{k-k_1}} \omega_{k-k_1} \\
&\leq \sum_{k \in \mathbb{Z}} \sum_{k_1 \in \mathbb{Z}} |a_{k_1}| |b_{k-k_1}| \omega_{k_1} \omega_{k-k_1} \\
&\leq \left(\sum_{k_1 \in \mathbb{Z}} |a_{k_1}| \omega_{k_1} \right) \left(\sum_{k_2 \in \mathbb{Z}} |b_{k_2}| \omega_{k_2} \right) \\
&= \|a\|_\omega \|b\|_\omega. \quad \square
\end{aligned}$$

Example 3.4. Consider a *geometric decay rate* $\nu \geq 1$ and an *algebraic decay rate* $s > 0$. Let $\omega = \{\omega_k\}_{k \in \mathbb{Z}}$ the sequence of positive real numbers defined component-wise by

$$\omega_k \stackrel{\text{def}}{=} (|k| + 1)^s \nu^{|k|}. \quad (14)$$

Then ω is an admissible sequence of weights. Indeed, for any $n, k \in \mathbb{Z}$,

$$\begin{aligned}
(\omega_{n+k})^{\frac{1}{s}} &= (|n+k| + 1) \nu^{\frac{|n+k|}{s}} \\
&\leq (|n| + |k| + 1) \nu^{\frac{|n|}{s}} \nu^{\frac{|k|}{s}} \\
&= \left((|n| + 1) \nu^{\frac{|n|}{s}} \right) \left((|k| + 1) \nu^{\frac{|k|}{s}} \right) \\
&= (\omega_n)^{\frac{1}{s}} (\omega_k)^{\frac{1}{s}} \\
&= (\omega_n \omega_k)^{\frac{1}{s}},
\end{aligned}$$

which implies that $\omega_{n+k} \leq \omega_n \omega_k$ since the function $f(x) \stackrel{\text{def}}{=} x^{\frac{1}{s}}$ is strictly monotone increasing on $(0, \infty)$ for any $s > 0$.

The following consequence of Lemma 3.3 allows to obtain rigorous theoretical bounds on any component of a discrete convolution.

Corollary 3.5. Fix $p \in \mathbb{N}$. Let $a^{(1)}, \dots, a^{(p)} \in \ell_\omega^1$. Then, for any $k \in \mathbb{Z}$

$$\left| \left(a^{(1)} * a^{(2)} * \dots * a^{(p)} \right)_k \right| \leq \|a^{(1)}\|_\omega \|a^{(2)}\|_\omega \dots \|a^{(p)}\|_\omega \frac{1}{\omega_k}. \quad (15)$$

Proof. Fix $k \in \mathbb{Z}$. Then by Lemma 3.3,

$$\begin{aligned}
\left| \left(a^{(1)} * a^{(2)} * \dots * a^{(p)} \right)_k \right| \omega_k &\leq \sum_{j \in \mathbb{Z}} \left| \left(a^{(1)} * a^{(2)} * \dots * a^{(p)} \right)_j \right| \omega_j \\
&= \|a^{(1)} * a^{(2)} * \dots * a^{(p)}\|_\omega \\
&\leq \|a^{(1)}\|_\omega \|a^{(2)}\|_\omega \dots \|a^{(p)}\|_\omega. \quad \square
\end{aligned}$$

4 Rigorous convolution enclosures: a refinement

We are now ready to present a refinement of the rigorous enclosures of the discrete convolutions presented in Section 2. Note that in practice, the method of Section 2 is essentially used for the finite part (i.e. $|k| < M$), while the method of Section 3 is essentially used for the tail part (i.e. for $M \leq |k| \leq p(M-1)$).

Combining the FFT algorithm with interval arithmetic (e.g. see the function `verifyfft.m` in INTLAB [23]), we can use formula (12) to compute a rigorous enclosure $\mathcal{B}_k^{(FFT)} \in \mathbb{C}$ such that

$$\sum_{\substack{k_1+k_2+\dots+k_p=k \\ |k_i|<M}} a_{k_1}^{(1)} a_{k_2}^{(2)} \dots a_{k_p}^{(p)} = \frac{1}{2pM-1} \cdot \mathcal{F}_k(\tilde{b}^{(1)} \hat{*} \dots \hat{*} \tilde{b}^{(p)}) \in \mathcal{B}_k^{(FFT)}, \quad (16)$$

for all $k = -p(M-1), \dots, p(M-1)$, where the vectors $\tilde{b}^{(1)}, \dots, \tilde{b}^{(p)}$ are defined in (9).

Now, choose an admissible sequence of weights $\omega \stackrel{\text{def}}{=} \{\omega_k\}_{k \in \mathbb{Z}}$. For all $k = -p(M-1), \dots, p(M-1)$, let

$$\beta_k \stackrel{\text{def}}{=} \|a^{(1)}\|_\omega \|a^{(2)}\|_\omega \dots \|a^{(p)}\|_\omega \omega_k^{-1}, \quad (17)$$

and set

$$\mathcal{B}_k^{(\omega)} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| \leq \beta_k\}. \quad (18)$$

From Corollary 3.5, we get that

$$\sum_{\substack{k_1+k_2+\dots+k_p=k \\ |k_i|<M}} a_{k_1}^{(1)} a_{k_2}^{(2)} \dots a_{k_p}^{(p)} \in \mathcal{B}_k^{(\omega)}. \quad (19)$$

Remark 4.1. We see from (19) that the quality of the enclosure of $\mathcal{B}_k^{(\omega)}$ depends on the choice of the admissible sequence of weights $\omega \stackrel{\text{def}}{=} \{\omega_k\}_{k \in \mathbb{Z}}$. In practice, we may know in advance that the type of functions we are approximating are analytic, in which case setting $s = 0$ and $\nu > 1$ in the weights (14) are most suitable. A least square fit can be used to determine the optimal geometric decay rate $\nu > 1$ fitting the data. Once this choice has been identified, set $\omega_k = \nu^{|k|}$, compute β_k using (17) and define $\mathcal{B}_k^{(\omega)}$. Similarly, we may know in advance that the type of functions we are approximating are only C^s , in which case setting $s > 0$ and $\nu = 1$ in the weights (14) are most suitable. A least square fit can be used to determine the optimal algebraic decay rate $s > 1$ fitting the data. Once this choice has been identified, set $\omega_k = (|k| + 1)^s$, compute β_k using (17) and define $\mathcal{B}_k^{(\omega)}$.

For each $k \in \mathbb{Z}$ such that $|k| \leq p(M-1)$, we refine the enclosure of the discrete convolution by setting

$$\mathcal{B}_k \stackrel{\text{def}}{=} \mathcal{B}_k^{(FFT)} \cap \mathcal{B}_k^{(\omega)}. \quad (20)$$

5 Examples

In this final section, we showcase how the refinement (20) of Section 4 helps stabilizing computing ℓ_ω^1 norms of discrete convolutions of the form (6). We present three application examples. We compute rigorously a cubic convolution (Section 5.1) and a convolution of

degree 20 (Section 5.2), and demonstrate how our new approach improves dramatically the standard method of Section 2. Finally in Section 5.4, we present a rigorous aposteriori error analysis for a steady state in the quintic Swift-Hohenberg PDE, and again compare the two approaches.

5.1 Cubic discrete convolutions

Let us consider a vector of Fourier/Chebyshev coefficients $\{a_k\}_{|k|<M}$ with some prescribed geometric decay rate $\rho > 1$. We want to compute a rigorous enclosure for cubic discrete convolutions of the form

$$(a^3)_k = (a * a * a)_k = \sum_{\substack{k_1+k_2+k_3=k \\ |k_i|<M}} a_{k_1} a_{k_2} a_{k_3}, \quad \text{for } |k| \leq 3(M-1). \quad (21)$$

Fix $p = 3$, $M = 30$ and $\rho = 4$. Consider a symmetric vector $\{\bar{a}_k\}_{|k|<M}$ such that $\bar{a}_{-k} = \bar{a}_k$ and $\bar{a}_k \in \mathbb{R}$. For the coefficients $\bar{a}_k$ ($k = 0, \dots, 29$), we consider $\bar{a}_k = \alpha_k \rho^{-k}$, where $\alpha_k \in [-1, 1]$ are chosen randomly. We computed rigorous component-wise bounds of $(\bar{a}^3)_k$ given in (21) for all $|k| \leq 3(M-1) = 87$ using the standard approach of Section 2 (rigorous FFT only) and the refinement of Section 4 (rigorous FFT intersected with the Banach algebra bounds). In Figure 1 on the left, we show the plot of both rigorous component-wise enclosure for $\log(|(\bar{a}^3)_k|)$ for $|k| \leq p(M-1) = 3(29) = 87$. In Table 1, we consider the interval vectors $\mathcal{B}^{(FFT)}$ and $\mathcal{B} \stackrel{\text{def}}{=} \mathcal{B}^{(FFT)} \cap \mathcal{B}^{(\omega)}$ (having each 87 interval components) and computed a rigorous upper bound for their norm in ℓ_ω^1 with $\omega = \{\nu^{|k|}\}_{|k| \leq 87}$ for different values of $\nu > 1$. We also compare the bounds with the theoretical upper bound $\|\bar{a}^3\|_\omega \leq \|\bar{a}\|_\omega^3$.

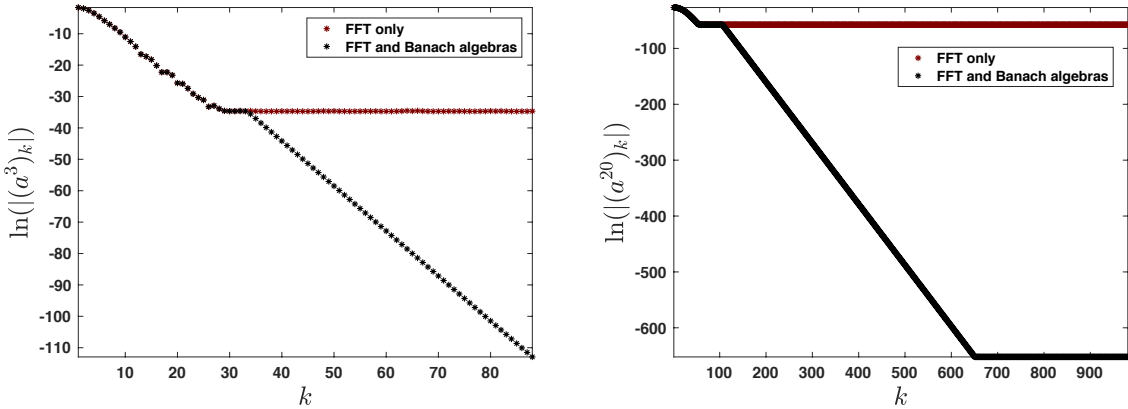


Figure 1: Two rigorous component-wise enclosure for: **(left)** $\log(|(\bar{a}^3)_k|)$ for $|k| \leq 87$; **(right)** $\log(|(\bar{a}^{20})_k|)$ for $|k| \leq 980$.

ν	$\ \mathcal{B}^{(FFT)}\ _\omega$	$\ \mathcal{B}^{(FFT)} \cap \mathcal{B}^{(\omega)}\ _\omega$	$\ \bar{a}\ _\omega^3$
1	9.9559×10^{-2}	9.9559×10^{-2}	1.6643×10^{-1}
1.25	1.8288×10^{-1}	1.8287×10^{-1}	3.9832×10^{-1}
1.5	2.5805×10^0	2.7271×10^{-1}	8.7256×10^{-1}
1.6	5.6391×10^2	2.7271×10^{-1}	1.1745×10^0
1.7	1.0032×10^5	4.0439×10^{-1}	1.5701×10^0
1.8	1.3436×10^7	4.9123×10^{-1}	2.0881×10^0
1.9	1.3926×10^9	5.9573×10^{-1}	2.7661×10^0
2	1.1450×10^{11}	7.2135×10^{-1}	3.6548×10^0
2.5	2.5847×10^{19}	1.8758×10^0	1.4778×10^1
3	1.8058×10^{26}	1.4199×10^1	6.8826×10^1
3.5	1.1272×10^{32}	2.8576×10^3	5.5624×10^2
4	1.1936×10^{37}	1.9558×10^6	2.8233×10^4

Table 1: Three different upper bounds for $\|\bar{a}^3\|_\omega$. In the first two columns, rigorous upper bounds for the ℓ_ω^1 norms of the interval vectors $\mathcal{B}^{(FFT)}$ and $\mathcal{B} \stackrel{\text{def}}{=} \mathcal{B}^{(FFT)} \cap \mathcal{B}^{(\omega)}$ (having each 87 interval components) with $\omega = \{\nu^{|k|}\}_{|k| \leq 87}$ for different values of $\nu \geq 1$. In the third column, we compare the bounds with the theoretical upper bound $\|\bar{a}^3\|_\omega \leq \|\bar{a}\|_\omega^3$.

5.2 Discrete convolutions of degree 20

We compute a rigorous enclosure for discrete convolutions of degree 20 of the form

$$(a^{20})_k = (\overbrace{a * \dots * a}^{20 \text{ times}})_k = \sum_{\substack{k_1 + \dots + k_{20} = k \\ |k_i| < M}} a_{k_1} \dots a_{k_{20}}, \quad \text{for } |k| \leq 20(M-1). \quad (22)$$

Fix $p = 20$, $M = 50$ and $\rho = 3$. Consider a symmetric vector $\{\bar{a}_k\}_{|k| < M}$ such that $\bar{a}_{-k} = \bar{a}_k$ and $\bar{a}_k \in \mathbb{R}$. For the coefficients $\bar{a}_k$ ($k = 0, \dots, 49$), we consider $\bar{a}_k = \alpha_k \rho^{-k}$, where $\alpha_k \in [-1/2, 1/2]$ are chosen randomly. We computed rigorous component-wise bounds of $(\bar{a}^{20})_k$ given in (22) for all $|k| \leq p(M-1) = 980$ using the standard approach of Section 2 and the refinement of Section 4. In Figure 1 on the right, we show the plot of both rigorous component-wise enclosure for $\log(|(\bar{a}^{20})_k|)$ for $|k| \leq p(M-1) = 980$. As in Section 5.1, we provide comparisons in Table 2.

ν	$\ \mathcal{B}^{(FFT)}\ _\omega$	$\ \mathcal{B}^{(FFT)} \cap \mathcal{B}^{(\omega)}\ _\omega$	$\ \bar{a}\ _\omega^{20}$
1	3.6095×10^{-11}	3.6095×10^{-11}	2.7979×10^{-2}
1.05	2.0193×10^{-3}	4.7327×10^{-11}	3.3327×10^{-2}
1.1	6.6628×10^{16}	6.3808×10^{-11}	3.9653×10^{-2}
1.2	3.9165×10^{53}	1.2718×10^{-10}	5.5972×10^{-2}
1.5	1.8339×10^{148}	2.0877×10^{-6}	1.5603×10^{-1}
1.7	2.7672×10^{201}	1.0043×10^0	3.1316×10^{-1}

Table 2: The three different upper bounds for $\|\bar{a}^{20}\|_\omega$.

5.3 Discrete convolutions of degree 100

We now compute a rigorous enclosure for discrete convolutions of degree 100 of the form

$$(a^{100})_k = (\overbrace{a * \dots * a}^{100 \text{ times}})_k = \sum_{\substack{k_1 + \dots + k_{100} = k \\ |k_i| < M}} a_{k_1} \dots a_{k_{100}}, \quad \text{for } |k| \leq 100(M-1). \quad (23)$$

Fix $p = 100$, $M = 11$ and $\rho = 30$. Consider a symmetric vector $\{\bar{a}_k\}_{|k| < M}$ such that $\bar{a}_{-k} = \bar{a}_k$ and $\bar{a}_k \in \mathbb{R}$. For the coefficients $\bar{a}_k$ ($k = 0, \dots, 1000$), we consider $\bar{a}_k = \alpha_k \rho^{-k}$, where $\alpha_k \in [-1, 1]$ are chosen randomly. We computed rigorous component-wise bounds of $(\bar{a}^{100})_k$ given in (22) for all $|k| \leq p(M-1) = 1000$ using the standard approach of Section 2 and the refinement of Section 4. We provide comparisons in Table 3.

ν	$\ \mathcal{B}^{(FFT)}\ _\omega$	$\ \mathcal{B}^{(FFT)} \cap \mathcal{B}^{(\omega)}\ _\omega$	$\ \bar{a}\ _\omega^{20}$
1	4.0676×10^{-31}	4.0676×10^{-31}	1.2570×10^{-1}
1.1	3.1261×10^{-2}	4.7431×10^{-31}	1.2812×10^{-1}
1.2	1.0476×10^{36}	5.5291×10^{-31}	1.3060×10^{-1}
1.3	4.3735×10^{70}	6.6089×10^{-31}	1.3315×10^{-1}
1.4	5.4031×10^{102}	2.4845×10^{-29}	1.3576×10^{-1}
1.5	4.2542×10^{132}	2.3256×10^{-26}	1.3845×10^{-1}
1.6	4.0393×10^{160}	1.0333×10^1	1.4121×10^{-1}

Table 3: The three different upper bounds for $\|\bar{a}^{100}\|_\omega$.

5.4 A posteriori error analysis in the quintic Swift-Hohenberg PDE

The quintic Swift-Hohenberg PDE (e.g. see [13, 24]) defined on a bounded interval with even periodic boundary conditions is given by

$$\begin{aligned} u_t &= (\lambda - 1)u - 2u_{xx} - u_{xxxx} + \mu u^3 - u^5, \quad \text{in } \Omega = [0, \frac{2\pi}{L}] \\ u(x, t) &= u(x + 2\pi/L, t), \quad u(x, t) = u(-x, t), \quad \text{on } \partial\Omega. \end{aligned} \quad (24)$$

The solutions can be expressed via the Fourier expansion

$$u(x, t) = \sum_{k=-\infty}^{\infty} a_k(t) e^{ikLx} = a_0 + 2 \sum_{k=1}^{\infty} a_k(t) \cos(kLx), \quad (25)$$

where $a_k \in \mathbb{R}$ and $a_{-k} = a_k$. Plugging (25) in (24) results in the infinite set of ODEs given by

$$\dot{a}_k = F_k(a) \stackrel{\text{def}}{=} \left(\lambda - (1 - k^2 L^2)^2 \right) a_k + \mu (a^3)_k - (a^5)_k, \quad (26)$$

where $a^3 = a * a * a$ and $a^5 = a * a * a * a * a$ are cubic and quintic discrete convolutions, respectively. Since $a_{-k} = a_k$, then $F_{-k} = F_k$. Looking for equilibria of the Swift-Hohenberg PDE (24) is equivalent to compute solutions of $F(a) = 0$ in ℓ_ω^1 , for some admissible sequence

of weights ω with geometric growth $\omega_k = \nu^{|k|}$ for some $\nu > 1$. As considered in [13], we consider the parameter values $L = 0.1$, $\mu = 3$ and $\lambda < 0$. In particular, we fix $\lambda = -1/2$. Equilibria $u = u(x)$ of (24) correspond to solutions of $F(a) = 0$, where $a = (a_k)_{k \in \mathbb{Z}}$ is the infinite sequence of Fourier coefficients and $F = (F_k)_{k \in \mathbb{Z}}$ is given component-wise by (26). We consider a Galerkin projection of (26) of dimension $M = 230$ and applied Newton's method to find a numerical approximation $\bar{a} = \{\bar{a}_k\}_{|k| < 230} \in \mathbb{R}^{459}$ ($\bar{a}_{-k} = \bar{a}_k$) such that $F(\bar{a}) \approx 0$. Applying a least square fit, we compute that the numerical solution $\bar{a}$ has a geometric decay rate of about 1.213, that is $|\bar{a}_k| \leq \frac{C}{1.213^{|k|}}$.

One of the approach in the field of rigorous numerics is called the *radii polynomial approach* [7, 15] and its goal is to demonstrate the existence (and local uniqueness) of solutions of some infinite dimensional nonlinear problems $F(a) = 0$ posed on a given Banach space X . The idea, based a Newton-Kantorovich type argument, is to prove that a certain Newton-like operator $T(a) = a - AF(a)$ is a contraction mapping theorem on a carefully chosen ball centered at a numerical approximation $\bar{a} \in X$, where A is an approximate inverse for $DF(\bar{a})$ in the sense that $\|I - ADF(\bar{a})\|_{B(X)} \ll 1$. In the works [9, 15, 26, 27, 28], the radii polynomial approach is applied on the Banach space ℓ_ω^1 , and in the process, the quantity $\|AF(\bar{a})\|_\omega$ has to be bounded rigorously. The bound for $\|AF(\bar{a})\|_\omega$ is often denoted by Y_0 . In Table 4, we present a list of rigorous upper bounds for $\|F(\bar{a})\|_\omega$ with $\omega = \{\omega_k\}_{k \in \mathbb{Z}} = \{\nu^{|k|}\}_{k \in \mathbb{Z}}$ for different values of $\nu \geq 1$, where the rigorous computation of $F(\bar{a})$ is performed using the two approaches presented in this paper (1. FFT alone as introduced in Section 2; and 2. FFT and Banach algebras as introduced in Section 4). We do a similar comparison for Y_0 satisfying $\|AF(\bar{a})\|_\omega \leq Y_0$ in Table 5. In these two tables, the refinement enclosure of Section 4 is a dramatic improvement over the standard approach. As $\|T(\bar{a}) - \bar{a}\|_\omega = \|AF(\bar{a})\|_\omega \leq Y_0$, the bounds presented in Table 5 are rigorous aposteriori error bound for a steady state in the quintic Swift-Hohenberg PDE (24).

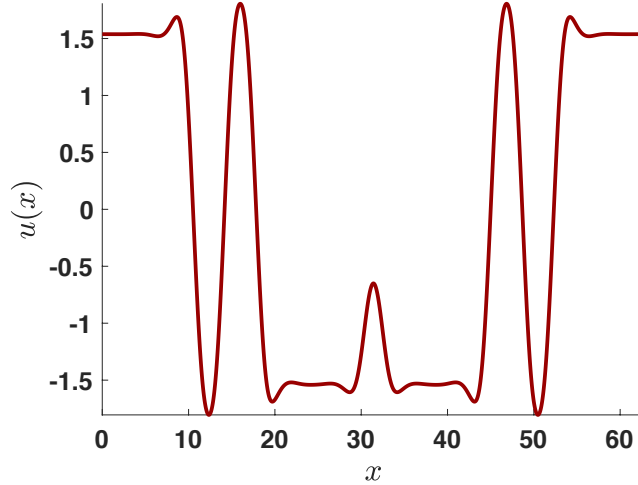


Figure 2: Steady state of the quintic Swift-Hohenberg equation (24) at parameter values $L = 0.1$, $\mu = 3$ and $\lambda = -1/2$. $M = 230$ Fourier coefficients are used to represent the solution.

ν	$\ F(\bar{a})\ _\omega$ (FFT only)	$\ F(\bar{a})\ _\omega$ (FFT + Banach algebras)
1	7.9453×10^{-10}	2.3107×10^{-10}
1.01	4.4067×10^{-6}	1.2869×10^{-9}
1.02	1.7514×10^{-1}	1.1573×10^{-8}
1.03	8.3728×10^3	1.3328×10^{-7}
1.04	4.0427×10^8	1.7290×10^{-6}
1.05	1.8729×10^{13}	2.3820×10^{-5}
1.06	8.1458×10^{17}	3.3840×10^{-4}
1.07	3.2908×10^{22}	4.8769×10^{-3}
1.08	1.2279×10^{27}	7.0628×10^{-2}
1.09	4.2188×10^{31}	1.0221×10^0

Table 4: Two different upper bounds for $\|F(\bar{a})\|_\omega$. In the first column, a rigorous upper FFT and interval arithmetic whereas in the second column, the bound is obtained using the refinement of Section 4.

ν	Y_0 (FFT only)	Y_0 (FFT + Banach algebras)
1	1.8068×10^{-6}	1.8068×10^{-6}
1.02	2.2227×10^{-6}	2.2227×10^{-6}
1.04	2.5844×10^0	2.7805×10^{-6}
1.05	1.1736×10^5	3.1380×10^{-6}
1.1	8.0385×10^{27}	2.7180×10^{-5}
1.11	2.3253×10^{32}	2.9418×10^{-4}
1.12	6.1836×10^{36}	3.9776×10^{-3}
1.13	1.5120×10^{41}	5.4766×10^{-2}

Table 5: Two different upper bounds for Y_0 satisfying $\|AF(\bar{a})\|_\omega \leq Y_0$. In the first column, a rigorous upper FFT and interval arithmetic whereas in the second column, the bound is obtained using the refinement of Section 4.

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