Computer assisted Fourier analysis in sequence spaces of varying regularity

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Abstract

This work treats a functional analytic framework for computer assisted Fourier analysis which can be used to obtain mathematically rigorous error bounds on numerical approximations of solutions of differential equations. An abstract a-posteriori theorem is employed in order to obtain existence and regularity results for C^k problems with $0 < k \leq \infty$ or $k = \omega$. The main tools are certain infinite sequence spaces of rapidly decaying coefficients: we employ sequence spaces of algebraic and exponential decay rates in order to characterize the regularity our results. We illustrate the implementation and effectiveness of the method in a variety of regularity classes. We also examine the effectiveness of spaces of algebraic decays for studying solutions of problems near the breakdown of analyticity.

Key words. Validated numerics, regularity classes, spatially inhomogeneous equations, Fourier analysis, breakdown of analyticity

1 Introduction and Background

Spectral methods are a fundamental tool in analysis. In particular they play an important role in the study of ordinary, delay, and partial differential equations. Part of the power of spectral methods is that they transform Banach spaces of smooth functions into Banach spaces of rapidly decaying sequences. This rapid decay is exploited in the development of numerical methods, as it justifies the approximation of an infinite sequence by a sequence of finite length. Numerical methods are especially useful for the insight they provide into nonlinear problems, where the presence of nonlinearities can frustrate classical pen and paper techniques.

In the last decades there has been considerable interest in techniques for mathematically rigorous a-posteriori validation of results of numerical computations. Such techniques go by various names including validated numerics, rigorous numerics, and computer assisted proof in analysis (though this is by no means a complete list). Indeed, these tools have turned the digital computer into a powerful tool for proving theorems in nonlinear analysis, as is illustrated for example by Lanford's proof of the Feigenbaum conjectures [1] or by Tucker's solution of Smale's 14-th problem [2, 3]. A thorough overview of the development of computer assisted analysis is beyond the scope of the present work, and we refer the reader to articles of [4, 5, 6, 7, 8] for more discussions of the literature. The interested reader may also want to consult the book of Tucker [9].

In the present work we fix our attention on computer assisted proofs utilizing spectral (in fact Fourier) methods, and study the interplay between the regularity of the equation and the

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choice of the Banach sequence space in which the computer assisted proof is formulated. The regularity of the equation of course governs the regularity of its solutions, hence determines the decay rate of the Fourier coefficients. We present an a-posteriori framework for computer assisted Fourier analysis which applies to large class of problems in a wide variety of regularity classes.

More precisely, we are interested in proving the existence of a zero of a smooth map $F\colon X \to Y$ between Banach spaces, and we are willing to accept computer assistance in order to achieve this goal. A sketch of a typical argument is as follows.

- **Step 1:** Project the problem of interest onto an appropriate Galerkin/spectral basis.
- **Step 2:** Truncate the problem and solve numerically.
- **Step 3:** Determine (or make an educated guess as to) the regularity of the true solution. Embed the numerical approximation into an appropriate sequence space which codifies this regularity class.
- **Step 4:** Prove a-posteriori that F has a true solution near the numerical approximation. The details of this proof will depend on the sequence space chosen in Step 3. In the present work this a-posteriori analysis employs a modified Newton-Kantorovich argument. Checking the hypotheses of the a-posteriori theorem involves a blend of analytic estimates and computer aided error bounds. These bounds are managed via a deliberate control of floating point errors, i.e using interval arithmetic.

In order to illustrate the utility of our approach we study non-trivial equilibrium solutions of a *spatially inhomogeneous Fisher equation*, i.e. we are interested in time constant solutions of

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + \mu u(t,x)(1 - c(x)u(t,x)), \qquad t \geq 0, \quad x \in [0,\pi]. \tag{1}$$

We refer to $c(x)$ as the *kernel function* for the problem, and note that by varying the smoothness of c we change the regularity of the desired solutions. Throughout we impose standard Neumann boundary conditions

$$\frac{\partial}{\partial x}u(t,0) = \frac{\partial}{\partial x}u(t,\pi) = 0, \qquad \text{for all} \quad t \geq 0.$$

The problem appears in population dynamics and mathematical ecology as a model of diffusive behavior in an inhomogeneous environment. We consider various choices of c with smoothness ranging from Lipschitz to C^∞ to real analytic, and discuss how the choice of c affects the formulation, implementation, and results of our computer assisted arguments.

We note that the present work deals with computer assisted analysis, rather than numerical analysis. In Section 3 for example we consider situations where we can obtain good numerical results (in the sense of small projected defects) regardless of the spatial inhomogeneity. However the computer assisted existence and regularity analysis succeeds in some sequence spaces and fails in others, again depending only on the smoothness of c.

Remark 1.1 (Choice of the example problem)**.** In addition to presenting a framework for mathematically rigorous computer assisted Fourier analysis, we also wish to discuss the use and performance of these techniques in varying regularity classes. Then we would like as much as possible to minimize technical complications, while highlighting features of the computer assisted arguments which depend only on regularity. These considerations inform our choice example problem. The one dimensional boundary value problem with quadratic nonlinearity and Neumann boundary conditions given by Equation (1) results in a particularly simple family

of sequence space maps, whose zeros we wish to study. At the same time the spatially inhomogeneous term allows for direct tuning of the smoothness of the problem, and determines which sequence spaces of rapidly decaying coefficients can be used.

Nevertheless, we do not want to give the false impression that the techniques discussed in the present work are limited to problems as simple as Equation (1). For example techniques related to those discussed here have been applied to problems with a higher number of spatial variables or problems involving space and time [10, 11, 12, 13, 14, 15, 16], to problems with non-polynomial/transcendental nonlinearities via techniques of interpolation/automatic-differentiation [17, 18, 19], to problems with nonlinearities involving function composition and renormalization [1, 20, 6], and to problems utilizing other spectral bases such as Chebyshev series [21, 22]. Again, the list is by no means exhaustive and is only mean to suggest the breadth of the field.

Remark 1.2 (Development of the $C^{\mathbf{k}}$ computer assisted Fourier analysis: convolution estimates)**.** The development of $C^{\mathbf{k}}$ convolution estimates in the field of computer assisted proofs in dynamics was originally motivated by the study of global dynamics of infinite dimensional dynamical systems (e.g. parabolic PDEs and infinite dimensional maps). In this context, a priori global bounds of the global attractor can sometimes be obtained in terms of Hilbert cubes in L^2 with algebraic decay rates. Once the a priori $C^{\mathbf{k}}$ bounds on the global attractor are obtained, the proofs of existence of the fixed points is done in the same $C^{\mathbf{k}}$ regularity class [10, 23, 24, 25]. Independently of the study of global dynamics, $C^{\mathbf{k}}$ convolution estimates were used to study bounded solutions of parabolic PDEs [26, 27, 28, 29]. Efforts in obtaining estimates for general polynomial nonlinearities have also been made [30, 25, 31, 30, 23].

Remark 1.3 (Analysis of functions which are $C^{\mathbf{k}}$ but not analytic)**.** If the solution of the problem at hand is analytic then computer assisted arguments in spaces of exponential decays, as pioneered by Lanford, will usually lead to excellent results. Moreover spaces of exponential decays have a well understood dual theory, and have natural Banach algebra structure which aids the nonlinear analysis. Because of these facts, computer assisted Fourier analysis for analytic problems is fairly elegant and elementary.

Analysis in sequence spaces of algebraic decays is more cumbersome. Dual space estimates are not available, the Banach algebra structure is less useful, and it is often necessary to develop special purpose convolution estimates. Of course not all problems have analytic solutions, and clearly there are situations where any formulation on a space of exponential decays is doomed to fail. Indeed the whole history of Fourier analysis seems to be driven by the need to study more and more irregular functions. We briefly suggest several possible applications which could benefit from lower regularity tools.

- In the biological, social, and more data driven physical and engineering sciences one often encounters models with parameters determined by experimental data. For example in some particular applications the spatially inhomogeneous term $c(x)$ in Equation (1), rather than being derived from first principles, may be the result of a best fit to experimental data. If piecewise linear, or cubic splines are used to provide the fit, then one would encounter kernels which are $C^{\mathbf{k}}$ rather than analytic. Here the spaces of algebraic decay would be appropriate.

- Results in the theory of state dependent delays [32] show that analytic delay equations can have solution which are only $C^{\mathbf{k}}$. If we are to study such problems by computer assisted Fourier analysis then spaces of algebraic decays could play a role.

- Sequence spaces of algebraic decay could be a useful tool for computer assisted Fourier analysis of problems on the verge of breakdown of analyticity. Such problems appear

naturally for example in the study of symplectic, Hamiltonian, and volume preserving dynamical systems. For these system quasi-periodic solutions typically appear in cantor-like arrangements, and it is observed that solutions decrease in regularity as we approach the boundary of the cantor set. The interested reader might consult for example the work of [33, 34, 35, 36] for more detailed discussion of these phenomena, as well as for analytical and numerical methods for their study.

Analysis of the quasi-periodic phenomena just mentioned requires sophisticated KAM techniques, and computer assisted proofs in this setting require delicate Nash-Moser arguments. In Section 3 we present a simplified model of the "breakdown of analyticity", and show that near the verge of breakdown computer assisted Fourier analysis in the sequence spaces of algebraic decay leads to more efficient computer assisted proofs than the analytic approach. This suggests that the spaces of algebraic decay could be useful for computer assisted Fourier/spectral analysis of problems near the breakdown of analyticity.

Remark 1.4 (Finite elements and Sobolev spaces)**.** Boundary value problems in dimension greater than one are often formulated on domains with complicated geometry, and in this setting spectral methods perform much more poorly. Finite-element methods are employed instead, and such methods do not lead to sequences of rapid decay. Rather error estimates are usually formulated in some classical Sobolev spaces. Computer assisted methods of proof in these settings are found in the work of [37, 38, 39, 40, 41, 42]. We remark that a-posteriori computer assisted proofs based on finite-element methods have a different flavour than the spectral methods discussed in the present work.

The remainder of the paper is organized as follows. In Section 1.1 we state the main a-posteriori theorem used in the present work. This theorem serves as the basis for all the computer assisted existence arguments to follow. In Section 1.2 we review some basic definitions and tools of Fourier analysis. In Sections 1.3.1, 1.3.2, 1.3.3, 1.3.4, and 1.3.5 we define the kernel functions $c(x)$ and in Sections 1.4 and 1.5 the sequence spaces studied in the remainder of the paper. In Section 2 we derive the bounds associated with our main example. These bounds are needed when we apply the tools of Section 1.1 to the Fisher equation in various regularity settings. Finally in Section 3 we discuss the results of a number of computer assisted proofs. All of the computer codes used to produce these results are freely available from [43]. For the sake of completeness we give some technical details and proofs in the Appendices.

1.1 A posteriori implicit function theory and Newton-Kantorovich analysis

Consider $F\colon X \to Y$ a map defined between the Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$. Denote by $B(X,Y)$ the set of bounded linear operator between X and Y. We simply write $B(X)$ to denote $B(X,X)$. Given a point $x \in X$ and a radius $r > 0$ denote respectively

$$B_r(x) \stackrel{\text{def}}{=} \{y \in X \mid \|y - x\|_X < r\} \qquad \text{and} \qquad \overline{B_r(x)} \stackrel{\text{def}}{=} \{y \in X \mid \|y - x\|_X \leq r\}$$

the open and the closed balls of radius r and centered at x in X.

The following theorem is useful for studying $F(x) = 0$ problems posed on X. The elementary proof is found in Appendix A.

Theorem 1.5. [A Newton-Kantorovich argument - the radii polynomial approach] *Consider bounded linear operators $A^\dagger \in B(X,Y)$ and $A \in B(Y,X)$. Assume $F\colon X \to Y$ is a*

C^1 Fréchet differentiable map, that A is injective and that

$$AF\colon X \to X. \tag{2}$$

Consider a point $\bar{x} \in X$ (typically a numerical approximation), and let Y_0, Z_0, Z_1, and Z_2 be nonnegative constants satisfying

$$\|AF(\bar{x})\|_X \leq Y_0 \tag{3}$$
$$\|I - AA^\dagger\|_{B(X)} \leq Z_0 \tag{4}$$
$$\|A[DF(\bar{x}) - A^\dagger]\|_{B(X)} \leq Z_1 \tag{5}$$
$$\|A[DF(b) - DF(\bar{x})]\|_{B(X)} \leq Z_2 r, \quad \textit{for all } b \in \overline{B_r(\bar{x})}. \tag{6}$$

Define the radii polynomial

$$p(r) \stackrel{\text{def}}{=} Z_2 r^2 - (1 - Z_1 - Z_0)r + Y_0. \tag{7}$$

If there exists $r_0 > 0$ such that

$$p(r_0) < 0,$$

then there exists a unique $\tilde{x}$ in the open ball $B_{r_0}(\bar{x})$ satisfying

$$F(\tilde{x}) = 0.$$

Remark 1.6. While our modification of the Newton-Kantorovich theorem is stated in the abstract, the theorem is actually crafted for a-posteriori analysis of a map F between Banach spaces of infinite sequences X and Y. As such, the theorem above differs somewhat from the usual Newotn-Kantorovich theorem (see for example [44]). In particular our Theorem requires the choice of two linear operators A and $A^\dagger$. The purpose of these operators is to remove the usual hypothesis of the existence of a bound on the inverse of the operator $DF(\bar{x})$, and to replace this condition with a hypotheses which closer to the sequence space truncation error analysis, and which are easily checked via numerical linear algebra and interval arithmetic.

The intuition is that A is an approximate left inverse of the operator $A^\dagger$, and that $A^\dagger$ is an approximation of $DF(\bar{x})$. In applications one chooses $A^\dagger$ by combining information about the derivative of the truncated problem with some information about the asymptotic spectrum of DF. In computer assisted proofs involving sequence spaces the operator $A^\dagger$ will be chosen to have a 2×2 block structure, where the upper left block is the Jacobian derivative of the Galerkin projection of DF, the lower right hand block is determined by the asymptotic eigenvalues of DF, and the remaining two blocks are zero maps.

Then A will have a similar 2×2 block structure where the upper left hand block is usually given by a numerical approximate inverse of the Jacobian of the Galerkin projection at the numerical solution. It is also required that A is a smoothing operator, in the sense that $AF(x), ADF(x)$ range in X. The explicit choice of these operators is a problem dependent affair, and is best illustrated through examples.

1.2 Review of Fourier series

Suppose that $f\colon [0, 2\pi] \to \mathbb{C}^N$ is a Lebesgue integrable function and define the *Fourier series* for f by

$$\sum_{n\in\mathbb{Z}} a_n e^{inx},$$

where the *Fourier coefficients* are defined by

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx,$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{inx}\,dx, \qquad 1 \leq |n| < \infty.$$

The convergence of the Fourier series to the function f (in various Banach spaces) is intrinsically related to the decay rate of the sequence of Fourier coefficients. We recall the following classical facts.

Theorem 1.7 (**Paley-Weiner results**). *Let $\{a_n\}_{n\in\mathbb{Z}}$ be a sequence of complex numbers and consider the formal Fourier series*

$$f(z) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} a_n e^{inz}.$$

- ***$C^{\mathbf{k}}$ functions:*** *For $s > 1$, let $\mathbf{k} \stackrel{\text{def}}{=} \lfloor s-1 \rfloor = \max\{k \in \mathbb{N} : k \leq s-1\}$. If*

  $$\sup_{n\in\mathbb{Z}} |a_n||n|^s < \infty,$$

  then the formal Fourier series defines a 2π-periodic function $\mathbf{k}$ times differentiable function on $[0, 2\pi]$.

- ***Analytic functions:*** If $\nu > 1$ has that

  $$\sum_{n\in\mathbb{Z}} |a_n|\nu^{|n|} < \infty,$$

  Then the formal Fourier series defines a 2π-periodic function of $[0, 2\pi]$ which extends analytically to the complex strip

  $$\mathcal{S}_\nu = \{z \in \mathbb{C} \ : |\text{imag}(z)| < \ln(\nu)\}.$$

  The function f is continuous on $\partial\mathcal{S}_\nu$. Moreover we have the following partial converse: if f is 2π-periodic and analytic on $\mathcal{S}_\nu$ then

  $$\sum_{n\in\mathbb{Z}} |a_n|\tau^{|n|} < \infty,$$

  for every $1 < \tau < \nu$, i.e. the Fourier series of f converges to absolutely and uniformly to f on any strip of width strictly less than $\ln(\nu)$.

We also have the following version of Merten's theorem for Fourier series.

Theorem 1.8 (Discrete convolution and pointwise multiplication of Fourier series)**.** *Suppose that*

$$f(x) = \sum_{k\in\mathbb{Z}} a_k e^{ikx}, \qquad \textit{and} \qquad g(x) = \sum_{k\in\mathbb{Z}} b_k e^{ikx},$$

converge pointwise, and that one of the coefficient sequences is absolutely summable. Then

$$(f \cdot g)(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx},$$

with

$$c_k = \sum_{k_1+k_2=k} a_{k_1} b_{k_2},$$

converges pointwise. The sequence $\{c_k\}$ *is referred to as the discrete convolution of the sequences* $\{a_k\}$ *and* $\{b_k\}$*.*

Remark 1.9. *Since in Equation* (1) *we impose Neumann boundary conditions, in fact we only have to deal with cosine series of the form*

$$f(x) = a_0 + 2\sum_{k=1}^{\infty} a_k \cos(kx),$$

which can also be thought of as a full complex Fourier series

$$f(x) = \sum_{k\in\mathbb{Z}} a_k e^{ikx},$$

with the symmetry condition $a_{-k} = a_k$*.*

1.3 Different kernel functions

We now give a brief description of the functions $c(x)$ which are studied in Section 3 as spatial inhomogeneities for Equation (1).

1.3.1 A continuous kernel function which is not Lipschitz continuous

Consider the kernel function $c(x) = c_0 + 2\sum_{n=1}^{\infty} c_n \cos(nx)$, with cosine Fourier coefficients given by

$$c_n = \begin{cases} 2, & n = 0 \\ \dfrac{-1}{n^{3/2}}, & n \geq 2 \text{ and } n \text{ even} \\ 0, & n \geq 2 \text{ and } n \text{ odd.} \end{cases} \tag{8}$$

A graph of the Kernel function $c(x)$ can be found in Figure 1.

1.3.2 C^0 piecewise linear Lipschitz bump function (spline)

Let $[x_0 - h, x_0 + h] \subset [0, \pi]$ and $D > 0$. Define the function

$$b^{(0)}(x) = \begin{cases} 0, & 0 \leq x \leq x_0 - h \\ \frac{D}{h}(x - x_0 + h), & x_0 - h < x \leq x_0 \\ -\frac{D}{h}(x - x_0 - h), & x_0 < x \leq x_0 + h \\ 0, & x_0 + h < x \leq \pi \end{cases} \tag{9}$$

We refer to $b^{(0)}(x)$ as a C^0 *bump function.*

The Fourier (cosine) coefficients are given by

$$b_0^{(0)} = \frac{1}{\pi}\int_0^{\pi} b^{(0)}(x)\,dx = \frac{Dh}{\pi},$$

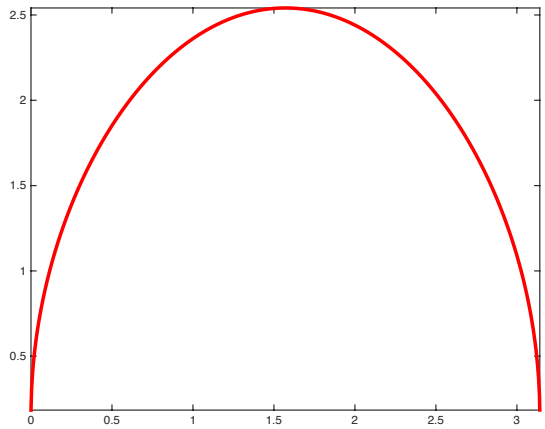

Figure 1: The kernel function $c(x)$ with Fourier coefficients given by (8) decaying as $n^{-\frac{3}{2}}$.

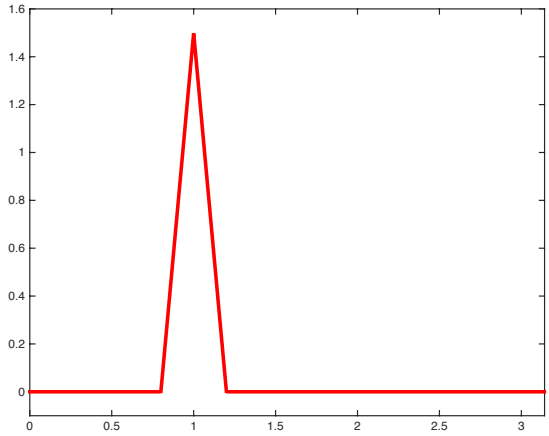

Figure 2: The C^0 spline/bump function defined in (9) with $x_0 = 1$, $h = 0.2$ and $D = 1.5$.

and

$$b_n^{(0)} = \frac{1}{\pi}\int_0^{\pi} b^{(0)}(x)\cos(nx)\,dx \tag{10}$$

$$= \frac{D}{\pi h}\int_{x_0-h}^{x_0}(x - x_0 + h)\cos(nx)\,dx - \frac{D}{\pi h}\int_{x_0}^{x_0+h}(x - x_0 - h)\cos(nx)\,dx \tag{11}$$

$$= \frac{2D}{\pi h n^2}\cos(nx_0)\left(1 - \cos(nh)\right). \tag{12}$$

1.3.3 C^2 piecewise cubic bump (B-spline)

Let $[x_0 - h, x_0 + h] \subset [0, \pi]$. Define the function

$$b^{(2)}(x) = \begin{cases} 0, & 0 \le x \le x_0 - 2h \\ \frac{1}{6}(2h + (x - x_0))^3, & x_0 - 2h < x \le x_0 - h \\ \frac{2h^3}{3} - \frac{1}{2}(x - x_0)^2(2h + (x - x_0)), & x_0 - h < x \le x_0 \\ \frac{2h^3}{3} - \frac{1}{2}(x - x_0)^2(2h - (x - x_0)), & x_0 < x \le x_0 + h \\ \frac{1}{6}(2h - (x - x_0))^3, & x_0 + h < x \le x_0 + 2h \\ 0, & x_0 + 2h < x \le \pi \end{cases} \tag{13}$$

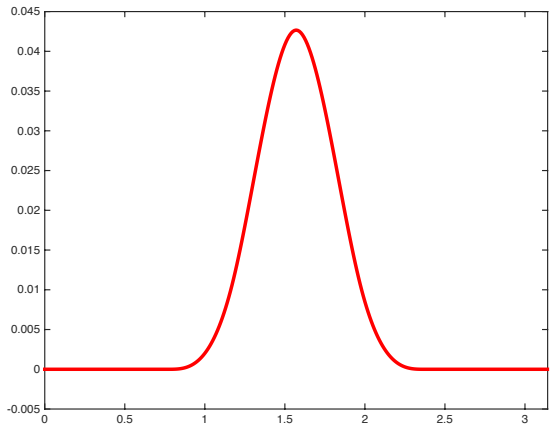

Figure 3: The piecewise cubic C^2 bump function given by (13) with $h = 0.4$ and $x_0 = \pi/2$.

The Fourier (cosine) coefficients are given by

$$b_0^{(2)} = \frac{1}{\pi} \int_0^\pi b^{(2)}(x)\, dx = \frac{h^4}{\pi}$$

and for $n \ge 1$,

$$b_n^{(2)} = \frac{1}{\pi} \int_0^\pi b^{(2)}(x) \cos(nx)\, dx = \frac{4}{\pi n^4} \cos(nx_0)(1 - \cos(nh))^2. \tag{14}$$

1.3.4 C^∞ function which is nowhere analytic

Define the set $\mathcal{S} = \{2^p : p = 0, 1, 2, 3, \dots\} = \{1, 2, 4, 8, 16, 32, \dots\}$, and consider the function

$$c(x) = c_0 + 2 \sum_{n \ge 1} c_n \cos(nx) = \sum_{n \in \mathcal{S}} e^{-\sqrt{n}} \cos(nx), \tag{15}$$

that is

$$c_n = \begin{cases} \frac{1}{2} e^{-\sqrt{n}}, & \text{if } n = 2^p \text{ for some } p \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

The function c is C^∞ but nowhere analytic on the interval $[0, \pi]$.

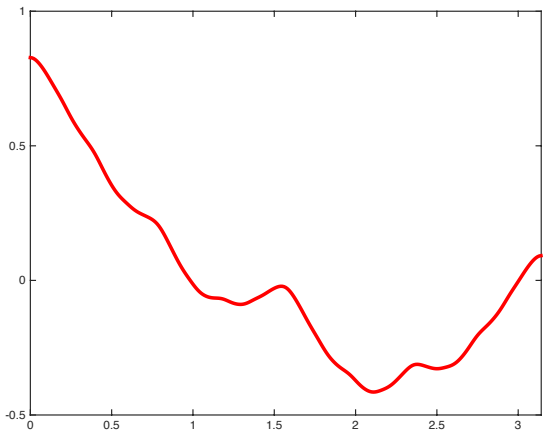

Figure 4: The C^∞ function (15) which is nowhere analytic.

1.3.5 Analytic "bump" function (not compactly supported)

In order to obtain an analytic analogue of the spike or bump functions discussed above we begin with the Poisson kernel

$$f_\sigma(x) = \frac{1-\sigma^2}{1-\sigma\cos(x)+\sigma^2} = 1 + 2\sum_{n=1}^{\infty} \sigma^n \cos(nx),$$

which we convolve with a less regular function with the desired profile. For example if we convolve with the C^0 bump function centered at x_0, of width $h > 0$ and of height $D > 0$, as defined in (9) in Section 1.3.2, we obtain the analytic bump function defined by

$$b^{(\omega)}(x) \stackrel{\text{def}}{=} (f_\sigma * b^{(0)})(x) = \int_0^\pi f_\sigma(s-x) b^{(0)}(s)\, ds, \tag{17}$$

which has Fourier cosine coefficients given by

$$b^{(\omega)}(x) = b_0^{(0)}(f_\sigma)_0 + 2\sum_{n=1}^{\infty} b_n^{(0)}(f_\sigma)_n \cos(nx).$$

In other words

$$b_0^{(\omega)} = b_0^{(0)}(f_\sigma)_0 = b_0^{(0)} \cdot 1 = \frac{1}{\pi}\int_0^\pi c(x)\, dx = \frac{Dh}{\pi},$$

and

$$b_n^{(\omega)} = b_n^{(0)}(f_\sigma)_n = \frac{2D}{\pi h n^2}\cos(nx_0)\,(1-\cos(nh))\,\sigma^n. \tag{18}$$

The resulting family of functions is illustrated in Figure 5.

Of course $b^{(\omega)}$ is not a true bump function (i.e. does not have compact support). However the function is analytic, does exhibit a unimodal "spike" at x_0 (the point about which $b^{(0)}(x)$ is centered), and does converge (in the L^2 sense) to $b^{(0)}$ as $\sigma \to 1$. As such $b^{(\omega)}$ could be used as the basis of an analytic interpolation scheme. Note that $b^{(\omega)}$ can be thought of as the evolution of $b^{(0)}$ under the heat equation (with Neumann boundary conditions).

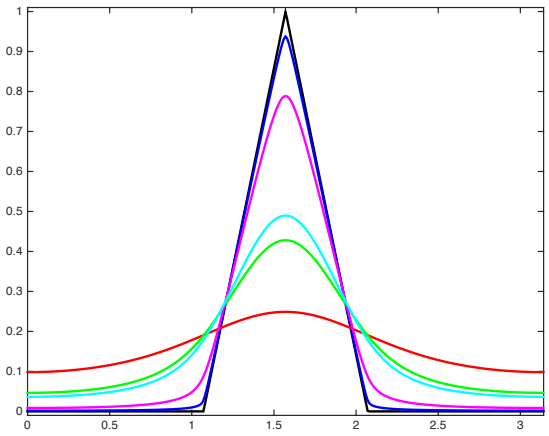

Figure 5: Analytic *bump* or spike functions obtained by convolution of the Poisson kernel with the C^0 bump function or tent map. The resulting analytic spikes are illustrated for $\sigma = 1$ i.e. the C^0 bump function itself (black), $\sigma = 0.99$ (blue), $\sigma = 0.95$ (magenta), $\sigma = 0.8$ (cayenne), $\sigma = 0.75$ (green), $\sigma = 0.5$ (red). Each of the convolved functions is analytic and satisfies the boundary conditions.

1.4 Exponential decay rates of Fourier coefficients and sequence spaces associated with analytic functions

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real (or complex) numbers. For any $\nu > 1$ define the norm

$$\|a\|_{1,\nu} \stackrel{\text{def}}{=} |a_0| + 2\sum_{n=1}^{\infty} |a_n|\nu^n,$$

and let

$$\ell^1_\nu \stackrel{\text{def}}{=} \{(a_n)_{n=0}^{\infty} \ : \ \|a\|_{1,\nu} < \infty\}. \tag{19}$$

The discrete cosine convolution product $*\colon \ell^1_\nu \times \ell^1_\nu \to \ell^1_\nu$ is defined component-wise by

$$(a * b)_n \stackrel{\text{def}}{=} \sum_{\substack{k_1+k_2=n \\ k_1,k_2\in\mathbb{Z}}} a_{|k_1|}b_{|k_2|} = \sum_{k=0}^{n} a_{n-k}b_k + \sum_{k=1}^{\infty}(a_{n+k}b_k + a_k b_{n+k}),$$

for any $a, b \in \ell^1_\nu$. Note that ℓ^1_ν is a Banach algebra under this product in the sense that

$$\|a * b\|_{1,\nu} \le \|a\|_{1,\nu}\|b\|_{1,\nu}.$$

In the sequel we are interested in finite dimensional truncations of sequences, operators, and mappings on ℓ^1_ν. The following notation is helpful. For $a \in \ell^1_\nu$ let $\pi_N\colon \ell^1_\nu \to \mathbb{R}^{N+1}$ be the canonical projection $\pi_N(a) = (a_0, \ldots, a_N)$ We employ the shorthand $a^{(N)} \stackrel{\text{def}}{=} \pi_N(a)$. Similarly let $\text{inc}\colon \mathbb{R}^{N+1} \to \ell^1_\nu$ denote the natural inclusion given by

$$\text{inc}(a^{(N)})_n = \begin{cases} a_n^{(N)}, & 0 \le n \le N \\ 0, & n \ge N+1. \end{cases}$$

When $a^{(N)} \in \mathbb{R}^{N+1}$ we abuse slightly the notation and write $\|a^{(N)}\|_{1,\nu}$ for

$$\|a^{(N)}\|_{1,\nu} = |a_0^{(N)}| + 2\sum_{n=1}^{N} |a_n^{(N)}|\nu^n = \|\mathrm{inc}\left(a^{(N)}\right)\|_{1,\nu}.$$

In other words we endow $\mathbb{R}^{N+1}$ with the subspace topology inherited from ℓ^1_ν. Similarly let $A^{(N)}$ be an $(N+1)\times(N+1)$ matrix. Again we abuse the notation and write $\|A^{(N)}\|$ in order to denote

$$\sup_{\|h\|_{1,\nu}=1} \|\mathrm{inc}(A^{(N)}h^{(N)})\|_{1,\nu},$$

and have the bound $\|A^{(N)}\| \le K_0 + K_N$ where K_0, K_N are as in Lemma 1.11.

In the sequel we make use of the dual of ℓ^1_ν. Let $\{b_n\}_{n=0}^\infty$ be a sequence of real (or complex) numbers and define the norm

$$\|b\|_{\infty,\nu^{-1}} \stackrel{\text{def}}{=} \max\left(|b_0|, \sup_{n\ge 1}\frac{|b_n|}{2\nu^n}\right).$$

Define

$$\ell^\infty_{\nu^{-1}} \stackrel{\text{def}}{=} \left\{(b_n)_{n=0}^\infty \,:\, \|b\|_{\infty,\nu^{-1}} < \infty\right\}.$$

The following Lemma relates $\ell^\infty_{\nu^{-1}}$ to $(\ell^1_\nu)^*$. In fact these are the same space (up to isomorphism).

Lemma 1.10 (**Duality for weighted ell-one spaces).**

$$\left(\ell^1_\nu\right)^* = \ell^\infty_{\nu^{-1}},$$

in the sense that these spaces are isometrically isomorphic.

The proof is classical and based on the fact that we can write down the isomorphism $\Phi\colon \ell^\infty_{\nu^{-1}} \to (\ell^1_\nu)^*$ as

$$\Phi[b](a) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} a_n b_n.$$

The isomorphism aids in obtaining bounds on linear functional and operators.

Given two Banach spaces X and Y, denote by $B(X,Y)$ the space of bounded linear operators from X to Y. If $X = Y$, then we simply denote $B(X,X)$ by $B(X)$. The following lemma is helpful for bounding the norm of a class of linear operators which appear frequently in applications involving numerics.

Lemma 1.11 (**Eventually diagonal linear operators).** *Let $A^{(N)}$ an $(N+1)\times(N+1)$ matrix and $\{\alpha_k\}_{k=N+1}^\infty$ a bounded sequence of numbers. Define the linear operator $A\colon \ell^1_\nu \to \ell^1_\nu$ by*

$$(Ah)_n \stackrel{\text{def}}{=} \begin{cases} (A^{(N)}h^{(N)})_n, & 0 \le n \le N \\ \alpha_n h_n, & n \ge N+1. \end{cases}$$

Let

$$\begin{aligned} K_0 &\stackrel{\text{def}}{=} \max\left(|a_{00}|, \max_{1\le k\le N}\frac{|a_{0k}|}{2\nu^k}\right) \\ K_N &\stackrel{\text{def}}{=} 2\sum_{n=1}^{N}\max\left(|a_{n0}|, \max_{1\le k\le N}\frac{|a_{nk}|}{2\nu^k}\right)\nu^n \\ K_\infty &\stackrel{\text{def}}{=} \sup_{n\ge N+1}|\alpha_n|. \end{aligned}$$

Then $A \in B(\ell^1_\nu)$ *and*

$$\|A\|_{B(\ell^1_\nu)} \le K_0 + K_N + K_\infty.$$

If $A^{(N)}$ *is invertible and* $\alpha_n \neq 0$ *for all* $n \ge N+1$ *then* A *is injective.*

The following technical lemma is the key to the truncation error analysis of the derivative of a nonlinear operator between sequence spaces.

Lemma 1.12 (Convolution sums for tails). *Let* $0 \le n \le N$ *and* $a^{(N)}, c^{(N)} \in \ell^1_\nu$ *be of the form* $a^{(N)} = (a_0, \dots, a_N, 0, 0, 0, \dots)$, $c^{(N)} = (c_0, \dots, c_N, 0, 0, 0, \dots)$. *For any* $h \in \ell^1_\nu$ *let* $h^{(\infty)} = (0, \dots, 0, h_{N+1}, h_{N+2}, \dots)$ *and define the linear functional* $l_n \colon \ell^1_\nu \to \mathbb{R}$ *by*

$$l_n(h) \stackrel{\text{def}}{=} (a^{(N)} * c^{(N)} * h^{(\infty)})_n.$$

Let

$$\alpha_n \stackrel{\text{def}}{=} \max_{N+1 \le k \le 2N-n} \frac{|(c^{(N)} * a^{(N)})_{n+k}|}{2\nu^k} \qquad \text{and} \qquad \beta_n \stackrel{\text{def}}{=} \max_{N+1 \le k \le 2N+n} \frac{|(c^{(N)} * a^{(N)})_{k-n}|}{2\nu^k}.$$

Then

$$\|l_n\|_{(\ell^1_\nu)^*} \le \alpha_n + \beta_n.$$

1.5 Algebraic decay rates of Fourier coefficients and sequence spaces associated $C^{\mathbf{k}}$ functions

Consider weights ω^q_n with algebraic growth rate $q > 1$ defined by

$$\omega^q_n \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } n = 0 \\ |n|^q, & \text{if } n \neq 0. \end{cases} \tag{20}$$

Given a sequence $a = (a_n)_{n \ge 0}$, define the norm

$$\|a\|_{\infty,q} \stackrel{\text{def}}{=} \sup_{n \ge 0} \{|a_n| \omega^q_n\}, \tag{21}$$

which is then used to define the Banach space

$$\ell^\infty_q = \{a = (a_n)_{n \ge 0} : \|a\|_{\infty,q} < \infty\}. \tag{22}$$

Analytic convolution estimates in the $C^{\mathbf{k}}$ category have been developed by several people [23, 26, 24, 10, 31, 11, 30]. The proof of the following essential estimate follows directly by Lemma B.3.

Lemma 1.13. *Fix an algebraic decay rate* $q > 1$. *Let* $a = (a_n)_{n \ge 0}, b = (b_n)_{n \ge 0}, c = (c_n)_{n \ge 0} \in \ell^\infty_q$. *Consider the definition of* $\alpha^{(3)}_n$ *for* $n \ge 0$ *as given in* (62). *Then, for any* $n \ge 0$,

$$|(a * b * c)_n| = \left| \sum_{\substack{n_1+n_2+n_3=n \\ n_1,n_2,n_3 \in \mathbb{Z}}} a_{|n_1|} b_{|n_2|} c_{|n_3|} \right| \le (\|a\|_{\infty,q} \|b\|_{\infty,q} \|c\|_{\infty,q}) \, \frac{\alpha^{(3)}_n}{\omega^q_n}. \tag{23}$$

2 A posteriori analysis of Equation (1) in varying regularity categories

We seek a non-constant equilibrium solution of Equation (1). We will assume that our kernel function $c(x)$ satisfies the boundary conditions, so that it can be written as

$$c(x) = c_0 + 2\sum_{n=1}^{\infty} c_n \cos(nx).$$

Then we look for an equilibrium solution of Equation (1) given by

$$u(x) = a_0 + 2\sum_{n=1}^{\infty} a_n \cos(nx).$$

Plugging the series expansion into Equation (1) and matching terms of like frequence reduces the problem to finding a solution of the infinite system of coupled algebraic equations

$$(\mu - n^2)a_n - \mu(c * a * a)_n = 0, \qquad n \geq 0. \tag{24}$$

The question is: in what Banach space (sequence space) do we look for the solution $\{a_n\}_{n=0}^{\infty}$? The question is determined by c, i.e. we will look for solution with coefficients $\{a_n\}_{n=0}^{\infty}$ having decay rate determined by the decay rate of the coefficients of the kernel. In particular we consider the following kernel functions.

- The function $c(x)$ is an analytic non compactly supported bump.
- The function $c(x)$ is C^{∞} but nowhere analytic.
- The function $c(x)$ is a C^2 piecewise cubic bump.
- The function $c(x)$ is a C^0 piecewise linear bump.
- The function $c(x)$ is continuous but not Lipschitz continuous.

It is convenient to reformulate as a zero finding problem, i.e. let $F\colon X \to Y$ be the map defined component-wise as

$$F_n(a) \stackrel{\text{def}}{=} (\mu - n^2)a_n - \mu(c * a * a)_n, \qquad n \geq 0. \tag{25}$$

An equilibrium of Equation (1) is equivalent to a zero Equation (25). We proceed for the moment without specifying X or Y, and develop some general results, without specifying the domain of the infinite dimensional map.

Define the linear operator $\mathcal{L}\colon X \to Y$ by

$$\mathcal{L}(a)_n = (\mu - n^2)a_n.$$

Assume that L is a bounded linear operator from X to Y. We write

$$F(a) = \mathcal{L}a - \mu(c * a * a).$$

Now, if $*\colon X \times X \to X$ is a binary operation on X and $(X, *)$ is a Banach algebra, then multiplication is differentiable. More explicitly consider the operator $N\colon X \to X$ defined by

$$N(a) = (a * a).$$

Then N is Fréchet differentiable and for any $h \in X$ we have

$$DN(a)h = 2(a * h).$$

Moreover since multiplication against a fixed $c \in X$ is a linear operation we have that the operator

$$N_c(a) := (c * a * a),$$

has

$$DN_c(a)h := 2(c * a * h).$$

So, F is Fréchet differentiable, and for any $a, h \in X$ the action of the differential $DF(a)\colon X \to Y$ on h is given by

$$DF(a)h = \mathcal{L}h - 2\mu(c * a * h). \tag{26}$$

The second derivative of the map is given by the bilinear form

$$D^2F(a)(u, v) = -2\mu(c * u * v).$$

Note then that $D^2F(a)$ is a bounded bilinear operator on ℓ^1_ν with

$$\|D^2F(a)\| \leq 2\mu\|c\|,$$

(where the norm is the bilinear operator norm in ℓ^1_ν). From this and the Mean Value Theorem follow the Lipschitz bound

$$\|DF(a) - DF(b)\| \leq 2\mu\|c\|_{1,\nu}\|a - b\|_{1,\nu}, \tag{27}$$

for any X. In fact the bound above follows directly from (26) (i.e. strictly speaking it is not necessary in this setting to appeal to the Mean Value Theorem).

Now we consider the truncated problem. We denote by $a^{(N)} = (a_0, \ldots, a_N)$ a vector in $\mathbb{R}^{N+1}$. Define the projection map $F^{(N)}\colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ by

$$F^{(N)}_n(a_0, \ldots, a_N) = (\mu - n^2)a_n - \mu(c^{(N)} * a^{(N)} * a^{(N)})^{(N)}_n, \tag{28}$$

where the operator $(c^{(N)} * \cdot * \cdot)^{(N)}\colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ is defined component-wise by

$$(c^{(N)} * a^{(N)} * a^{(N)})^{(N)}_n \stackrel{\text{def}}{=} \sum_{\substack{k_1+k_2+k_3=n \\ |k_1|,|k_2|,|k_3|\leq N}} c_{|k_1|}a_{|k_2|}a_{|k_3|},$$

for $0 \leq n \leq N$. Note that $F^{(N)}(a^{(N)}) = \pi_N\left(F\left(\mathrm{inc}(a^{(N)})\right)\right)$, where recall that

$$\mathrm{inc}(a^{(N)})_n = \begin{cases} a^{(N)}_n, & 0 \leq n \leq N \\ 0, & n \geq N+1. \end{cases}$$

We also define the truncation map $(c * \cdot * \cdot)^{(\infty)}\colon X \to \mathbb{R}^N$ component-wise by

$$(c * a * a)^{(\infty)}_n \stackrel{\text{def}}{=} (c * a * a)_n - (c^{(N)} * a^{(N)} * a^{(N)})^{(N)}_n = \sum_{\substack{n_1+n_2+n_3=n \\ |n_1| \text{ or } |n_2| \text{ or } |n_3|>N}} c_{|n_1|}a_{|n_2|}a_{|n_3|}.$$

Note the if $\bar{a}$ has the property that $a_n = 0$ when $n \geq N+1$ and c^∞ has the property that $c^\infty_n = 0$ when $n \leq N$ then

$$(c^\infty * \bar{a} * \bar{a})^{(\infty)}_n = \sum_{\substack{n_1+n_2+n_3=n \\ |n_1|,|n_2|\leq N \text{ and } |n_3|>N}} c^\infty_{|n_1|}\bar{a}_{|n_2|}\bar{a}_{|n_3|}.$$

Then for such $\bar{a}$ we have

$$(c * \bar{a} * \bar{a})_n = (c^{(N)} * a^{(N)} * a^{(N)})_n^N + (c^\infty * \bar{a} * \bar{a})_n^\infty.$$

We think of both $(c^{(N)} * \cdot * \cdot)^{(N)}$ and $(c * \cdot * \cdot)^{(\infty)}$ as operators of one variable, i.e. the same vector or sequence is entered in both "empty slots" making these quadratic terms in $a^{(N)}$ and a respectively.

Using this notation we have that for $0 \leq n \leq N$ the mapping F has the decomposition

$$F_n(a) = F_n^{(N)}(a^{(N)}) - \mu(c * a * a)_n^{(\infty)}.$$

Suppose now that $\bar{a}^{(N)} = (\bar{a}_0, \ldots, \bar{a}_N) \in \mathbb{R}^N$ is an approximate solution of the $F(a) = 0$ problem, i.e. suppose that $\|F^{(N)}(\bar{a}^{(N)})\| \ll 1$. Let $\bar{a} \stackrel{\text{def}}{=} \text{inc}(\bar{a}^{(N)}) = (\bar{a}_0, \ldots, \bar{a}_N, 0, 0, 0, 0, 0, 0)$. We abuse notation by identifying $\bar{a}$ with $\bar{a}^{(N)}$ depending on the context. Assume that $DF^{(N)}(\bar{a})$ is invertible and let $A^{(N)}$ be an approximate inverse of $DF^{(N)}(\bar{a})$, i.e. suppose that $A^{(N)}$ is an $(N+1) \times (N+1)$ matrix having that $\|I - A^{(N)} DF^{(N)}(\bar{a})\| \ll 1$.

We now define the linear operators $A^\dagger$ and A in terms of component-wise action on h given by the formulas

$$(A^\dagger h)_n \stackrel{\text{def}}{=} \begin{cases} \left(DF^{(N)}(\bar{a}) h^{(N)}\right)_n & \text{for } 0 \leq n \leq N \\ (\mu - n^2) h_n & \text{for } n > N \end{cases} \tag{29}$$

and

$$(Ah)_n \stackrel{\text{def}}{=} \begin{cases} \left(A^{(N)} h^{(N)}\right)_n & \text{for } 0 \leq n \leq N \\ (\mu - n^2)^{-1} h_n & \text{for } n > N. \end{cases} \tag{30}$$

If

$$N + 1 > \sqrt{\mu} \tag{31}$$

holds, then $|\mu - n^2| \neq 0$ for all $n > N$. It follows that A and $A^\dagger$ are surjective, as the matrices $DF^{(N)}(\bar{a})$ and $A^{(N)}$ are invertible and the diagonal "tails" of A and $A^\dagger$ are non-zero.

Note that we now have all the ingredients necessary for Theorem 1.5. In the remainder of the section we show how to obtain estimates of the form required by that Theorem in several Banach sequence spaces X.

2.1 The radii polynomial approach for Fisher in the analytic category

Consider the kernel function having Fourier/cosine coefficients $c \in \ell^1_\nu$, and let $1 < \nu' < \nu$. Then the map F as defined by (25) satisfies $F \colon \ell^1_\nu \to \ell^1_{\nu'}$. We now give conditions sufficient to apply Theorem 1.5.

Throughout the remainder of the Section we assume that $N + 1 > \sqrt{\mu}$.

2.1.1 The Y_0 bound

Define

$$Y_0 \stackrel{\text{def}}{=} \|A^{(N)} F^{(N)}(\bar{a})\|_{1,\nu} + \mu \|A^{(N)}\| \|\bar{a}\|_{1,\nu}^2 \|c^{(\infty)}\|_{1,\nu} + \mu \frac{\|c^{(\infty)}\|_{1,\nu} \|\bar{a}\|_{1,\nu}^2}{(N+1)^2 - \mu} + 2 \sum_{n=N+1}^{3N} \mu \frac{|(c^{(N)} * \bar{a} * \bar{a})_n|}{n^2 - \mu} \nu^n.$$

In order to see that Y_0 so defined satisfies the hypotheses Theorem 1.5 we write

$$F_n(\bar{a}) = \begin{cases} F_n^{(N)}(\bar{a}) - \mu(c * \bar{a} * \bar{a})_n^{(\infty)} & \text{if } 0 \leq n \leq N \\ -\mu(c * \bar{a} * \bar{a})_n & \text{if } n \geq N + 1 \end{cases}$$

as $\bar{a}_n = 0$, hence $(\mu - n^2)\bar{a}_n = 0$ for all $n \geq N+1$. Then

$$AF(\bar{a})_n = \begin{cases} \left(A^{(N)}F^{(N)}(\bar{a}) - \mu A^{(N)}(c * \bar{a} * \bar{a})^{(\infty)}\right)_n & \text{if } 0 \leq n \leq N \\ -\dfrac{\mu(c * \bar{a} * \bar{a})_n}{\mu - n^2} & \text{if } n \geq N+1 \end{cases}.$$

Since $\bar{a}_n = 0$ for $n \geq N+1$ we have that

$$\begin{aligned} A^{(N)}(c * \bar{a} * \bar{a})^{(\infty)} &= A^{(N)}\pi_N\left((c^{(\infty)} * \bar{a} * \bar{a})\right) \\ &= A^{(N)}(c^{(\infty)} * \bar{a} * \bar{a})^{(N)}, \end{aligned}$$

Then

$$\begin{aligned} \left\|A^{(N)}(c * \bar{a} * \bar{a})^{(\infty)}\right\|_{1,\nu} &\leq \left\|A^{(N)}(c^{(\infty)} * \bar{a} * \bar{a})^{(N)}\right\|_{1,\nu} \\ &\leq \mu\|A^{(N)}\|\|\bar{a}\|_{1,\nu}^2\|c^{(\infty)}\|_{1,\nu}. \end{aligned}$$

Similarly for any $n \geq 0$ we write

$$(c * \bar{a} * \bar{a})_n = (c^{(N)} * \bar{a} * \bar{a})_n + (c^{(\infty)} * \bar{a} * \bar{a})_n.$$

Then

$$\begin{aligned} 2\sum_{n=N+1}^{\infty}\mu\frac{|(c * \bar{a} * \bar{a})_n|}{|\mu - n^2|}\nu^n &= 2\sum_{n=N+1}^{\infty}\mu\frac{\left|(c^{(N)} * \bar{a} * \bar{a})_n\right|}{|\mu - n^2|}\nu^n + 2\mu\sum_{n=N+1}^{\infty}\frac{\left|(c^{(\infty)} * \bar{a} * \bar{a})_n\right|}{|\mu - n^2|}\nu^n \\ &\leq 2\sum_{n=N+1}^{3N}\mu\frac{\left|(c^{(N)} * \bar{a} * \bar{a})_n\right|}{|\mu - n^2|}\nu^n + \mu\frac{\|c^{(\infty)}\|_{1,\nu}\|\bar{a}\|_{1,\nu}^2}{(N+1)^2 - \mu}. \end{aligned}$$

Combining the observations above leads to

$$\|AF(\bar{a})\|_{1,\nu} \leq \left\|A^{(N)}F^{(N)}(\bar{a})\right\|_{1,\nu} + \mu\left\|A^{(N)}(c * \bar{a} * \bar{a})^{(\infty)}\right\|_{1,\nu} + 2\sum_{n=N+1}^{\infty}\mu\frac{|(c * \bar{a} * \bar{a})_n|}{|\mu - n^2|}\nu^n \leq Y_0,$$

as desired.

2.1.2 The Z_0 bound

We have that

$$\|I - AA^\dagger\| = \|I - A^{(N)}DF^{(N)}(\bar{a})\| \leq Z_0,$$

simply by considering the definitions of A and $A^\dagger$. In particular the tail of $AA^\dagger$ is the identity and is completely canceled by the tail of I.

2.1.3 The Z_1 bound

Define the constants

$$\alpha_k \stackrel{\text{def}}{=} \max_{N+1 \leq j \leq 2N-k}\frac{|(c^{(N)} * \bar{a})_{j+k}|}{2\nu^j}, \qquad \text{and} \qquad \beta_k \stackrel{\text{def}}{=} \max_{N+1 \leq j \leq 2N+k}\frac{|(c^{(N)} * \bar{a})_{j-k}|}{2\nu^j},$$

and let a_{ij} denote the entries of the $(N+1)\times(N+1)$ matrix $A^{(N)}$. Then define

$$Z_1 \stackrel{\text{def}}{=} 2\mu\sum_{k=0}^{N}|a_{0k}^{(N)}|(\alpha_k+\beta_k)+4\mu\sum_{n=1}^{N}\sum_{k=0}^{N}|a_{nk}^{(N)}|(\alpha_k+\beta_k)\nu^n+\frac{2\mu}{(N+1)^2-\mu}\|c\|_{1,\nu}\|\bar{a}\|_{1,\nu},$$

In order to show that Z_1 so defined satisfies the hypotheses of Theorem 1.5 we write

$$\begin{aligned}(DF(\bar{a})h)_n &= (\mu-n^2)h_n-2\mu(c*\bar{a}*h)_n\\ &= \begin{cases}\left(DF^{(N)}(\bar{a})h^{(N)}\right)_n-2\mu(c*\bar{a}*h)_n^{(\infty)}, & \text{if } 0\le n\le N\\ (\mu-n^2)h_n-2\mu(c*\bar{a}*h)_n, & \text{if } n\ge N+1\end{cases}.\end{aligned}$$

Then

$$\left[(A^\dagger-DF(\bar{a}))h\right]_n=\begin{cases}-2\mu(c*\bar{a}*h)_n^{(\infty)} & \text{if } 0\le n\le N\\ -2\mu(c*\bar{a}*h)_n & \text{if } n\ge N+1,\end{cases}$$

and

$$\left[A(A^\dagger-DF(\bar{a}))h\right]_n=\begin{cases}-2\mu\left(A^{(N)}(c*\bar{a}*h)^{(\infty)}\right)_n, & \text{if } 0\le n\le N\\ \dfrac{-2\mu(c*\bar{a}*h)_n}{(\mu-n^2)}, & \text{if } n\ge N+1\end{cases}.$$

Now let $h\in\ell^1_\nu$ have unit norm and note that

$$\left\|A(A^\dagger-DF(\bar{a}))h\right\|_{1,\nu}=2\mu\left\|A^{(N)}(c*\bar{a}*h)^{(\infty)}\right\|_{1,\nu}+2\sum_{n=N+1}^{\infty}2\mu\frac{|(c*\bar{a}*h)_n|}{|\mu-n^2|}\nu^n\le Z_1.$$

Since

$$A^{(N)}(c^{(N)}*\bar{a}*h)^{(\infty)}=A^{(N)}\begin{pmatrix}(c^{(N)}*\bar{a}*h^{(\infty)})_0\\ \vdots\\ (c^{(N)}*\bar{a}*h^{(\infty)})_N\end{pmatrix},$$

we have

$$[A^{(N)}(c^{(N)}*\bar{a}*h)^{(\infty)}]_n=\sum_{k=0}^{N}a_{nk}(c^{(N)}*\bar{a}*h^{(\infty)})_k.$$

Applying Lemma 1.12 gives

$$\begin{aligned}2\mu\left\|A^{(N)}(c^{(N)}*\bar{a}*h)^{(\infty)}\right\|_{1,\nu} &\le 2\mu\left|[A^{(N)}(c^{(N)}*\bar{a}*h)^{(\infty)}]_0\right|+4\mu\sum_{n=1}^{N}\left|[A^{(N)}(c^{(N)}*\bar{a}*h)^{(\infty)}]_n\right|\nu^n\\ &\le 2\mu\sum_{k=0}^{N}|a_{0k}^{(N)}|(\alpha_k+\beta_k)+4\mu\sum_{n=1}^{N}\sum_{k=0}^{N}|a_{nk}^{(N)}|(\alpha_k+\beta_k)\nu^n,\end{aligned}$$

as desired.

For the tail term we apply the Banach algebra property and have that

$$2\sum_{n=N+1}^{\infty}2\mu\frac{|(c*\bar{a}*h)_n|}{|\mu-n^2|}\nu^n\le\frac{2\mu}{(N+1)^2-\mu}\|c\|_{1,\nu}\|\bar{a}\|_{1,\nu},$$

when $\|h\|=1$.

2.1.4 The Z_2 bound

We obtain that

$$Z_2 = 2\mu\|A\|_{B(\ell^1_\nu)}\|c\|_{1,\nu},$$

directly from Equation (27).

2.2 The radii polynomial approach for Fisher in the $C^{\mathbf{k}}$ category

In this section, we develop the general bounds Y_0, Z_0, Z_1, and Z_2 required to construct the radii polynomial as defined in Theorem 1.5 in the category of $C^{\mathbf{k}}$ functions.

Throughout the rest of the section, we consider a computational parameter

$$M > 2N. \tag{32}$$

which allows doing more computations to reduce the use of analytic estimates to obtain the bounds.

2.2.1 The Y_0 bound

Recall the definition of the bound Y_0 satisfying (3). In our case, the Banach space is $X = \ell^\infty_q$ and so Y_0 satisfies

$$\|T(\bar{a}) - \bar{a}\|_{\infty,q} = \|AF(\bar{a})\|_{\infty,q} \leq Y_0.$$

To facilitate the computation of the bound Y_0, we begin by constructing component-wise bounds $(Y_0^{(n)})_{n\geq 0}$ satisfying

$$|[T(\bar{a}) - \bar{a}]_n| = |[AF(\bar{a})]_n| \leq Y_0^{(n)}, \qquad \forall\ n \geq 0. \tag{33}$$

Set $Y_F = (Y_0, Y_1, \ldots, Y_N) \in \mathbb{R}^{N+1}$ to be

$$Y_0^{(n)} = |[A^{(N)}F^{(N)}(\bar{a})]_n|, \qquad \text{for } n = 0, \ldots, N. \tag{34}$$

Now, for $N+1 \leq n \leq M-1$, set

$$Y_0^{(n)} = \frac{\mu}{n^2 - \mu}|(c * \bar{a} * \bar{a})_n|, \tag{35}$$

since in this case, $F_n(\bar{a}) = (\mu - n^2)\bar{a}_n - \mu(c * \bar{a} * \bar{a})_n = -\mu(c * \bar{a} * \bar{a})_n$.

Remark 2.1 (Computing the finite $Y_0^{(n)}$bounds with FFT). *For each $0 \leq n \leq M-1$, the sum $(c * \bar{a} * \bar{a})_n$ consists of only finitely many terms. Indeed, for $0 \leq n \leq M-1$*

$$(c * \bar{a} * \bar{a})_n = \sum_{\substack{n_1+n_2+n_3=n \\ |n_1|\leq 2N+M-1 \\ |n_2|,|n_3|\leq N}} c_{n_1}\bar{a}_{n_2}\bar{a}_{n_3},$$

which is a finite convolution sum. This means that the computation of the bounds (34) *and* (35) *can be computed using interval arithmetic and the FFT algorithm.*

We look for an asymptotic bound $\tilde{Y}_M$ such that, for all $n \geq M$,

$$Y_0^{(n)} = \frac{\tilde{Y}_0^{(M)}}{\omega_n^q}. \tag{36}$$

Using the estimates of Lemma 1.13, it follows that for all $n \geq M$,

$$\begin{aligned}|[T(\bar{a}) - \bar{a}]_n| &= \frac{\mu}{n^2 - \mu}|(c * \bar{a} * \bar{a})_n| \\ &\leq \frac{\mu}{n^2 - \mu}\left(\|c\|_{\infty,q}\|\bar{a}\|^2_{\infty,q}\right)\frac{\alpha^{(3)}_n}{\omega^q_n} \\ &\leq \left[\left(\frac{\mu}{M^2 - \mu}\right)\alpha^{(3)}_M\|c\|_{\infty,q}\|\bar{a}\|^2_{\infty,q}\right]\frac{1}{\omega^q_n},\end{aligned}$$

and so we can set

$$\tilde{Y}^{(M)}_0 = \left(\frac{\mu}{M^2 - \mu}\right)\alpha^{(3)}_M\|c\|_{\infty,q}\|\bar{a}\|^2_{\infty,q}. \tag{37}$$

Note that the asymptotic bound (37) can be improved depending on the specific decaying properties of $c = \{c_n\}_{n\geq 0}$. In the examples we consider later in the paper, we will improve this bound.

Combining (34), (35) and (36),

$$\|AF(\bar{a})\|_{\infty,q} \leq Y_0 \stackrel{\text{def}}{=} \sup_{n\geq 0}\left\{Y^{(n)}_0\omega^q_n\right\} = \max\left\{\max_{n=0,\ldots,M-1}\{Y^{(n)}_0\omega^q_n\}, \tilde{Y}^{(M)}_0\right\}. \tag{38}$$

2.2.2 The Z_0 bound

Recall the definition of A in (30) and of $A^\dagger$ in (29). Let $\omega^{-q}_F \stackrel{\text{def}}{=} (\omega^{-q}_n)^N_{n=0} \in \mathbb{R}^{N+1}$ and let

$$Z_0 \stackrel{\text{def}}{=} \max_{n=0,\ldots,N}\left\{\left(\left|I - A^{(N)}DF^{(N)}(\bar{a})\right|\omega^{-q}_F\right)_n\omega^q_n\right\}. \tag{39}$$

Then

$$\|I - AA^\dagger\|_{B(\ell^\infty_q)} = \sup_{\|v\|_{\infty,q}\leq 1}\|(I - AA^\dagger)v\|_{\infty,q} \leq Z_0.$$

2.2.3 The Z_1 bound

We look for a bound Z_1 such that

$$\|A[DF(\bar{a}) - A^\dagger]\|_{B(\ell^\infty_q)} \leq Z_1.$$

Consider $h \in B_1(0)$. Then,

$$\left|\left((DF(\bar{a}) - A^\dagger)h\right)_n\right| \leq \begin{cases}2\mu(|\bar{a}| * |c| * \omega^{(\infty)})_n, & 0 \leq n \leq N \\ 2\mu(|\bar{a}| * |c| * \omega)_n, & n > N,\end{cases} \tag{40}$$

where $\omega \stackrel{\text{def}}{=} \{\omega^{-q}_n\}_{n\geq 0}$ and

$$\omega^{(\infty)} \stackrel{\text{def}}{=} \begin{cases}0, & \text{if } 0 \leq n \leq N \\ n^{-q}, & \text{if } n > N.\end{cases}$$

Consider C_M such that

$$|c_n| \leq C_M, \quad \text{for all } n \geq M. \tag{41}$$

For $n \in \{0, \dots, N\}$,

$$\begin{aligned}
2\mu(|\bar{a}| * |c| * \omega^{(\infty)})_n &\le 2\mu \sum_{\substack{n_1+n_2+n_3=n \\ |n_1|,|n_2|<M}} |\bar{a}_{n_1}||c_{n_2}|\omega^{(\infty)}_{n_3} + 2\mu \sum_{\substack{n_1+n_2+n_3=n \\ |n_1|<M\le|n_2|}} |\bar{a}_{n_1}||c_{n_2}|\omega^{(\infty)}_{n_3} \\
&\le 2\mu S^{(1)}_n + 2\mu C_M \left(\sum_{n_1=-N}^{N} |\bar{a}_{n_1}| \right) \left(\sum_{|n_3|=N+1}^{\infty} \frac{1}{|n_3|^q} \right) \\
&\le z^{(1)}_n \stackrel{\text{def}}{=} 2\mu S^{(1)}_n + 4\mu C_M \left(\sum_{n_1=-N}^{N} |\bar{a}_{n_1}| \right) \left(\sum_{n_3=N+1}^{M-1} \frac{1}{n_3^q} + \frac{1}{(q-1)M^{q-1}} \right), \quad (42)
\end{aligned}$$

where

$$S^{(1)}_n \stackrel{\text{def}}{=} \sum_{\substack{n_1+n_2+n_3=k \\ |n_1|,|n_2|<M}} |\bar{a}_{n_1}||n_{k_2}|\omega^{(\infty)}_{n_3}$$

is a finite sum and can be evaluated using the FFT algorithm and interval arithmetic. Similarly, for $n \in \{N+1, \dots, M-1\}$,

$$\begin{aligned}
2\mu\,|(\bar{a} * c * \omega)_n| &\le 2\mu \sum_{\substack{n_1+n_2+n_3=n \\ |n_1|,|n_2|<M}} |\bar{a}_{n_1}||c_{n_2}|\omega_{n_3} + 2\mu \sum_{\substack{n_1+n_2+n_3=n \\ |n_1|<M\le|n_2|}} |\bar{a}_{n_1}||c_{n_2}|\omega_{n_3} \\
&\le z^{(1)}_n \stackrel{\text{def}}{=} 2\mu S^{(2)}_n + 2\mu C_M \left(\sum_{n_1=-N}^{N} |\bar{a}_{n_1}| \right) \left(1 + 2\sum_{n_3=1}^{M-1} \frac{1}{n_3^q} + \frac{2}{(q-1)M^{q-1}} \right), \quad (43)
\end{aligned}$$

where

$$S^{(2)}_n \stackrel{\text{def}}{=} \sum_{\substack{n_1+n_2+n_3=n \\ |n_1|,|n_2|<M}} |\bar{a}_{n_1}||c_{n_2}|\omega_{n_3}$$

is a finite sum and can be evaluated using the FFT algorithm and interval arithmetic. For $n \ge M$, we use Lemma 1.13 to get

$$2\mu(|\bar{a}| * |c| * \omega)_n \le z^{(1)}_n \stackrel{\text{def}}{=} 2\mu \|\bar{a}\|_{\infty,q} \|c\|_{\infty,q} \alpha^{(3)}_M \frac{1}{\omega^q_n}, \quad (44)$$

where

$$\alpha^{(3)}_M \stackrel{\text{def}}{=} \alpha^{(2)}_M \sum_{k_1=1}^{M} \frac{1}{k_1^q} + \frac{2\alpha^{(2)}_M}{M^{q-1}(q-1)} + \Sigma^* + \sum_{k_1=1}^{M} \frac{\alpha^{(2)}_{k_1}}{k_1^q} + \alpha^{(2)}_M + \alpha^{(2)}_0.$$

Recalling (42), let $z^{(1)}_F \stackrel{\text{def}}{=} (z^{(1)}_0, z^{(1)}_1, \dots, z^{(1)}_N) \in \mathbb{R}^{N+1}$. Recalling (42) and (44), set

$$Z^{(n)}_1 \stackrel{\text{def}}{=} \begin{cases} \left(A^{(N)} z^{(1)}_F \right)_n, & n = 0, \dots, N \\ \dfrac{1}{n^2 - \mu} z^{(1)}_n, & n = N+1, \dots, M-1 \\ \left(\dfrac{2\mu}{M^2 - \mu} \|\bar{a}\|_{\infty,q} \|c\|_{\infty,q} \alpha^{(3)}_M \right) \dfrac{1}{\omega^q_n}, & n \ge M. \end{cases}$$

Hence, we get that

$$\begin{aligned}\|A[DF(\bar{a})-A^\dagger]\|_{B(\ell^\infty_q)} &= \sup_{\|h\|_{\infty,q}\le 1} \left\|A(DF(\bar{a})-A^\dagger)h\right\|_{\infty,q}\\ &\le \sup_{n\ge 0}\left\{Z_1^{(n)}\omega_n^q\right\}\\ &= Z_1 \stackrel{\text{def}}{=} \max\left(\max_{n=0,\dots,M-1}\left\{Z_1^{(n)}\omega_n^q\right\}, \frac{2\mu}{M^2-\mu}\|\bar{a}\|_{\infty,q}\|c\|_{\infty,q}\alpha_M^{(3)}\right). \end{aligned} \tag{45}$$

2.2.4 The Z_2 bound

We look for a bound Z_2 such that

$$\|A[DF(a)-DF(\bar{a})]\|_{B(\ell^\infty_q)} \le Z_2 r, \quad \text{for all } a\in B_r(\bar{a}).$$

Let $h\in B_1(0)$. Then,

$$(DF(a)-DF(\bar{a}))h = -2\mu(c*(a-\bar{a})*h).$$

Since $\|h\|_{\infty,q}\le 1$ and $\|a-\bar{a}\|_{\infty,q}\le r$, we get from Lemma 1.13, that for any $n\ge 0$,

$$|2\mu(c*(a-\bar{a})*h)_n| \le z_n^{(2)} r \stackrel{\text{def}}{=} \left(2\mu\|c\|_{\infty,q}\alpha_n^{(3)}\frac{1}{\omega_n^q}\right) r. \tag{46}$$

Recalling (46), let $z_F^{(2)} \stackrel{\text{def}}{=} (z_0^{(2)}, z_1^{(2)}, \dots, z_N^{(2)}) \in \mathbb{R}^{N+1}$. Set

$$Z_2^{(n)} \stackrel{\text{def}}{=} \begin{cases} \left(A^{(N)} z_F^{(2)}\right)_n, & n=0,\dots,N\\ \dfrac{1}{n^2-\mu} z_n^{(2)}, & n=N+1,\dots,M-1\\ \left(\dfrac{2\mu}{M^2-\mu}\|c\|_{\infty,q}\alpha_M^{(3)}\right)\dfrac{1}{\omega_n^q}, & n\ge M. \end{cases}$$

Hence, we get that

$$\begin{aligned}\|A[DF(a)-DF(\bar{a})]\|_{B(\ell^\infty_q)} &= \sup_{\|h\|_{\infty,q}\le 1} \|A[DF(a)-DF(\bar{a})]h\|_{\infty,q}\\ &\le \sup_{n\ge 0}\left\{Z_2^{(n)}\omega_n^q\right\}\\ &= Z_2 \stackrel{\text{def}}{=} \max\left(\max_{n=0,\dots,M-1}\left\{Z_2^{(n)}\omega_n^q\right\}, \frac{2\mu}{M^2-\mu}\|c\|_{\infty,q}\alpha_M^{(3)}\right). \end{aligned} \tag{47}$$

Combining the bounds Y_0, Z_0, Z_1 and Z_2 given respectively by (38), (39), (45) and (47), we have all the bounds to define the radii polynomial $p(r)$ as defined in (7).

3 Numerical Results and Comparisons

3.1 Computer-assisted proofs for Fisher in the analytic category

In this section, we present some computer-assisted proofs in the analytic category. Consider the analytic bump function (not compactly supported) as considered in Section 1.3.5 with Fourier coefficients given by

$$c_n = \begin{cases} \dfrac{Dh}{\pi}, & n = 0 \\ \dfrac{2D\sigma^n}{\pi h n^2}\cos(nx_0)\left(1-\cos(nh)\right), & n \geq 1. \end{cases} \tag{48}$$

Denote $\rho = 1/\sigma > 1$.

Theorem 3.1. *For each point on Figure 6 the rigorous verification method was successful in proving the existence of a unique equilibria of* (1) *near the numerically computed solution.*

Proof. Fix $\nu = 1.01$. The proof is obtained by running the script `script_analytic_proof_ex1_th1.m` which computes with interval arithmetic the coefficients of the radii polynomial $p(r)$ as defined in Section 2.1. For each numerically computed solution $\bar{x}$ in the MATLAB data files `ex1_analyticBump_branchj` with `j` $\in \{1,2,3,4,5\}$, the code verifies the existence of an interval $\mathcal{I} = (r_{\min}, r_{\max})$ such that for each $r_0 \in \mathcal{I}$, $p(r_0) < 0$, and so by Theorem 1.5, there exists a unique $\tilde{a} \in B_{r_0}(\bar{a}) \in \ell^1_\nu$ such that $F(\tilde{a}) = 0$ with F given component-wise in (25). The function $\tilde{u}(x) \stackrel{\text{def}}{=} \tilde{a}_0 + 2\sum_{n=1}^{\infty} \tilde{a}_n \cos(nx)$ is an equilibrium solution of (1). □

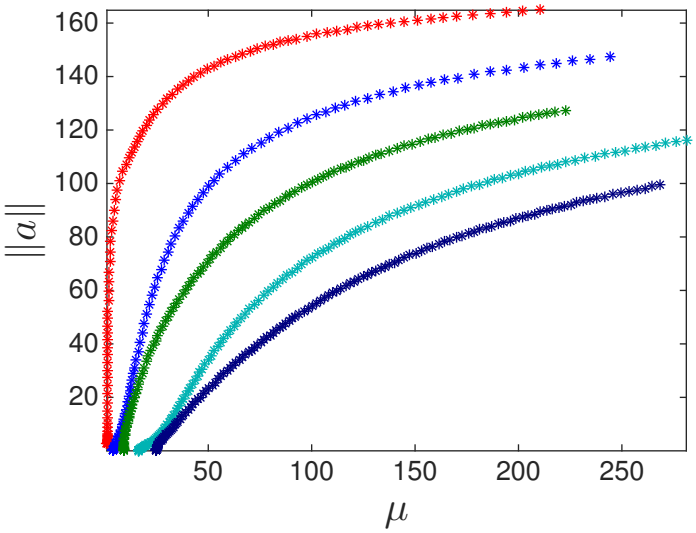

Figure 6: Five different branches of equilibrium solutions of the Fisher equation with the analytic bump kernel function with Fourier coefficients given by (48). In this case $h = 0.1$, $D = 1$, $x_0 = 1$ and $\rho = 1.1$. All points on this graph have been rigorously verified in the analytic category in Theorem 3.1 and in the $C^{\mathbf{k}}$ categoryTheorem 3.2.

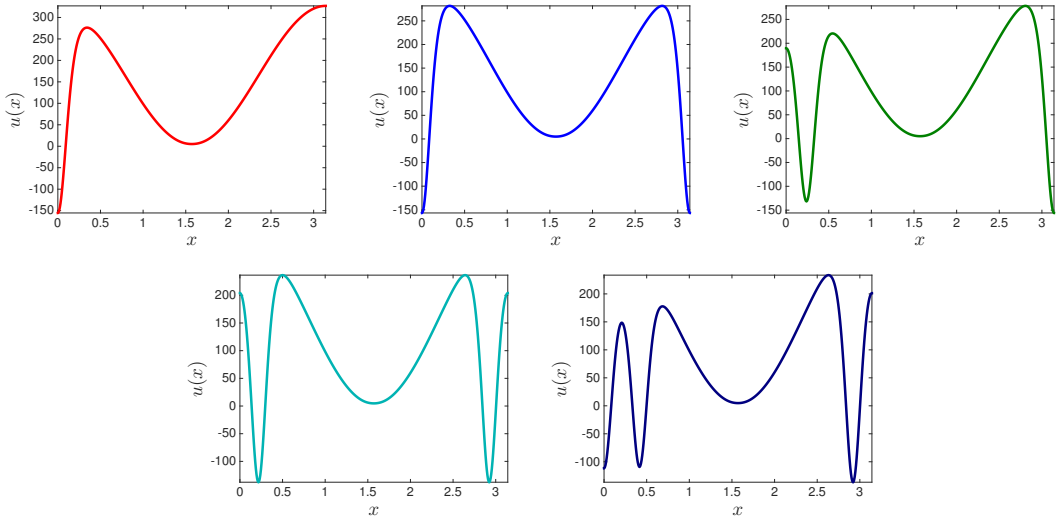

Figure 7: The most right points of each branch of Figure 6 as computed in Theorem 3.1 and Theorem 3.2. From top left to bottom, the points are labeled 1,2,3,4 and 5 respectively. The error bounds in ℓ^1_ν and in in ℓ^∞_q for each point are given in Figure 8.

label	$\mathcal{I}_{C^{\mathbf{k}}}$	$\mathcal{I}_{C^\omega}$
1	$[3.5495 \times 10^{-9}, 0.033837]$	$[1.9847 \times 10^{-6}, 0.22934]$
2	$[3.409 \times 10^{-9}, 0.030059]$	$[1.842 \times 10^{-6}, 0.23586]$
3	$[1.0374 \times 10^{-8}, 0.0080319]$	$[1.3988 \times 10^{-6}, 0.074376]$
4	$[1.6258 \times 10^{-8}, 0.0047913]$	$[1.5002 \times 10^{-6}, 0.059457]$
5	$[1.4368 \times 10^{-8}, 0.00408]$	$[1.1569 \times 10^{-6}, 0.054742]$

Figure 8: Different data for the proofs with the analytic *bump* kernel function with Fourier coefficients given by (48). For all computations, we utilized 500 Fourier modes, and we fixed $D = 1.5$, $x_0 = 1$ and $h = 0.2$. For the proofs in the $C^{\mathbf{k}}$ category, we used $q = 1.3$ and $M = 2000$. The interval on which the radii polynomial is positive is denoted by $\mathcal{I}_{C^{\mathbf{k}}}$. For the proofs in the analytic category, we fixed $\nu = 1.01$. The interval on which the radii polynomial is positive is denoted by $\mathcal{I}_{C^\omega}$.

3.2 Computer-assisted proofs for Fisher in the $C^{\mathbf{k}}$ category

In this section, we consider the five kernels considered in Section 1.3, and for each of them, we present some computer-assisted proofs. For each of the examples, we complete the construction of the required bounds to construct the radii polynomial. Note that the bounds Y_0, Z_0 and Z_2 were obtained in full generality in Section 2.2.1, Section 2.2.2 and Section 2.2.4, respectively. However, as mentioned in Section 2.2.1, the asymptotic bound (37) can be improved depending on the specific decaying properties of $c = \{c_n\}_{n\geq 0}$. For some of the examples, we present such an improvement. The only missing part is the Z_1 bound, which is also defined generally in (45), but its definition depends on the constant C_M satisfying (41) which is problem dependent.

3.2.1 The analytic "bump" kernel function

Note that for all $n \geq 1$, the analytic "bump" kernel function given by (48) satisfies

$$|c_n| \leq \frac{4D}{\pi h n^2 \rho^n}$$

and so, the bound C_M satisfying (41) is given by

$$C_M \stackrel{\text{def}}{=} \frac{4D}{\pi h M^2 \rho^M}.$$

If $n \geq M > 2N$ and $|n_1|, |n_2| \leq N$, then $n - n_1 - n_2 \geq M - 2N > 0$. In this case,

$$|c_{n-n_1-n_2}| \leq \frac{4D}{\pi h (n - n_1 - n_2)^2 \rho^{n-n_1-n_2}} \leq \frac{4D}{\pi h (M - 2N)^2 \rho^{n-n_1-n_2}}.$$

Given $q > 1$ and $\rho > 1$, the function $x \mapsto \frac{x^q}{\rho^x}$ is decreasing for $x \geq \frac{q}{\ln \rho}$. Indeed, it has a derivative given by $h(x)\,(q - x \ln \rho)$ with $q - x \ln \rho < 0 < h(x)$ for all $x > \frac{q}{\ln \rho}$. Hence, assuming that

$$M \geq \frac{q}{\ln \rho}, \tag{49}$$

we get that for all $n \geq M$,

$$\begin{aligned}
|(c * \bar{a} * \bar{a})_n| &= \left| \sum_{n_1+n_2+n_3=n} \bar{a}_{n_1} \bar{a}_{n_2} c_{n_3} \right| \\
&\leq \sum_{n_1=-N}^{N} \sum_{n_2=-N}^{N} |\bar{a}_{n_1}||\bar{a}_{n_2}||c_{n-n_1-n_2}| \\
&\leq \frac{4D}{\pi h} \left(\sum_{n_1=-N}^{N} \sum_{n_2=-N}^{N} \frac{n^q}{(n - n_1 - n_2)^2 \rho^{n-n_1-n_2}} |\bar{a}_{n_1}||\bar{a}_{n_2}| \right) \frac{1}{n^q} \\
&\leq \frac{4DM^q}{\pi h (M - 2N)^2 \rho^{M-2N}} \left(\sum_{j=-N}^{N} |\bar{a}_j| \right)^2 \frac{1}{n^q}.
\end{aligned}$$

Hence, recalling (37), set

$$\tilde{Y}_0^{(M)} \stackrel{\text{def}}{=} \left(\frac{\mu}{M^2 - \mu} \right) \min \left\{ \alpha_M^{(3)} \|c\|_{\infty,q} \|\bar{a}\|_{\infty,q}^2 \,,\; \frac{4DM^q}{\pi h (M - 2N)^2 \rho^{M-2N}} \left(\sum_{j=-N}^{N} |\bar{a}_j| \right)^2 \right\}. \tag{50}$$

Using (50), we can define Y_0 given in (38).

Combining the above value of C_M and the value of $\tilde{Y}_0^{(M)}$ given by (50) with the general bounds Y_0, Z_0, Z_1 and Z_2 given respectively by (38), (39), (45) and (47), we have all the bounds to define the radii polynomial $p(r)$ as defined in (7). The following result is proved using the radii polynomial approach.

Theorem 3.2. *For each point on Figure 6 the rigorous verification method was successful in proving the existence of a unique equilibria of* (1) *near the numerically computed solution.*

Proof. The proof is obtained by running the script `script_proof_ex1_th1.m` which computes with interval arithmetic the coefficients of the radii polynomial $p(r)$ in (7). For each numerically computed solution $\bar{x}$ in the MATLAB data files `ex1_analyticBump_branchj` with `j` $\in \{1,2,3,4,5\}$, the code verifies the existence of an interval $\mathcal{I} = (r_{\min}, r_{\max})$ such that for each $r_0 \in \mathcal{I}$, $p(r_0) < 0$, and so by Theorem 1.5, there exists a unique $\tilde{a} \in B_{r_0}(\bar{a})$ such that $F(\tilde{a}) = 0$ with F given component-wise in (25). The function $\tilde{u}(x) \stackrel{\text{def}}{=} \tilde{a}_0 + 2\sum_{n=1}^{\infty} \tilde{a}_n \cos(nx)$ is an equilibrium solution of (1). □

3.2.2 The C^∞ but nowhere analytic kernel function

Consider the kernel C^∞ function which is nowhere analytic as described in Section 1.3.4 with Fourier coefficients given by (16). In this case,

$$|c_n| \leq \frac{1}{2e^{\sqrt{n}}}, \qquad \text{for all } n \geq 1,$$

and so, the bound C_M satisfying (41) is given by

$$C_M \stackrel{\text{def}}{=} \frac{1}{2e^{\sqrt{M}}}, \qquad \text{for all } n \geq M.$$

If $n \geq M > 2N$ and $|n_1|, |n_2| \leq N$, then $n - n_1 - n_2 > 0$. In this case, $|c_{n-n_1-n_2}| \leq \frac{1}{2e^{\sqrt{n-n_1-n_2}}}$. If the algebraic decay rate $q > 1$ is chosen such that

$$q^2 < 2N, \tag{51}$$

then the function $x \mapsto \frac{x^q}{2e^{\sqrt{x-2N}}}$ is strictly decreasing for $x > 2N$. Indeed, it has a derivative given by $h(x)\left(2q\sqrt{x-2N} - x\right)$ with $2q\sqrt{x-2N} - x < 0 < h(x)$ for all $x > 2N$. Hence, for all $n \geq M$,

$$\begin{aligned}
|(c * \bar{a} * \bar{a})_n| &= \left| \sum_{n_1+n_2+n_3=n} \bar{a}_{n_1} \bar{a}_{n_2} c_{n_3} \right| \\
&\leq \sum_{n_1=-N}^{N} \sum_{n_2=-N}^{N} |\bar{a}_{n_1}||\bar{a}_{n_2}||c_{n-n_1-n_2}| \\
&\leq \left(\frac{n^q}{2e^{\sqrt{n-2N}}} \sum_{n_1=-N}^{N} \sum_{n_2=-N}^{N} |\bar{a}_{n_1}||\bar{a}_{n_2}| \right) \frac{1}{n^q} \\
&\leq \left(\frac{M^q}{2e^{\sqrt{M-2N}}} \right) \left(\sum_{j=-N}^{N} |\bar{a}_j| \right)^2 \frac{1}{\omega_n^q}.
\end{aligned}$$

Hence, recalling (37), set

$$\tilde{Y}_0^{(M)} \stackrel{\text{def}}{=} \left(\frac{\mu}{M^2 - \mu} \right) \min \left\{ \alpha_M^{(3)} \|c\|_{\infty,q} \|\bar{a}\|_{\infty,q}^2 \, , \; \frac{M^q}{2e^{\sqrt{M-2N}}} \left(\sum_{j=-N}^{N} |\bar{a}_j| \right)^2 \right\}. \tag{52}$$

Combining the above value of C_M and the value of $\tilde{Y}_0^{(M)}$ given by (52) with the general bounds Y_0, Z_0, Z_1 and Z_2 given respectively by (38), (39), (45) and (47), we have all the bounds to define the radii polynomial $p(r)$ as defined in (7). The following results are proved using the radii polynomial approach.

Theorem 3.3. *For each point on Figure 9 the rigorous verification method was successful in proving the existence of a unique equilibria of* (1) *near the numerically computed solution.*

Proof. The proof is obtained by running the script `script_proof_ex2_th1.m` which computes with interval arithmetic the coefficients of the radii polynomial $p(r)$ in (7). For each numerically computed solution $\bar{x}$ in the MATLAB data files `ex2_smooth_non_analytic_branchj` with `j` $\in \{1, 2, 3\}$, the code verifies the existence of an interval $\mathcal{I} = (r_{\min}, r_{\max})$ such that for each $r_0 \in \mathcal{I}$, $p(r_0) < 0$, and so by Theorem 1.5, there exists a unique $\tilde{a} \in B_{r_0}(\bar{a})$ such that $F(\tilde{a}) = 0$ with F given component-wise in (25). The function $\tilde{u}(x) \stackrel{\text{def}}{=} \tilde{a}_0 + 2\sum_{n=1}^{\infty} \tilde{a}_n \cos(nx)$ is an equilibrium solution of (1). □

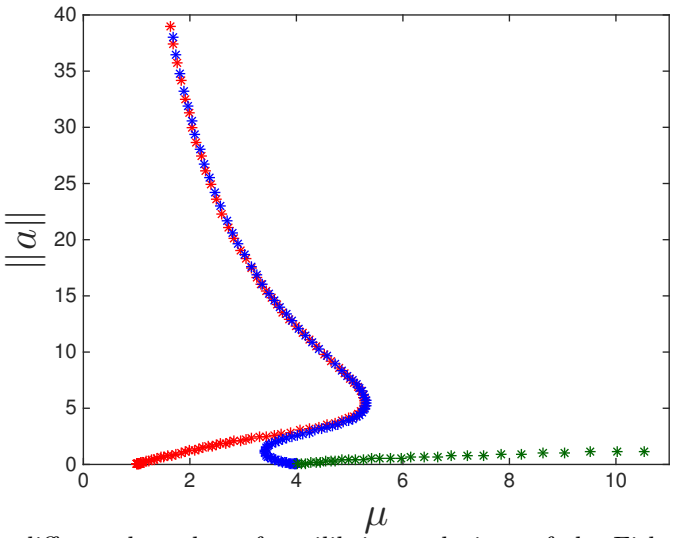

Figure 9: Three different branches of equilibrium solutions of the Fisher equation with the C^∞ but nowhere analytic kernel function (15). All points on this graph have been rigorously verified in Theorem 3.3.

Theorem 3.4. *At $\mu = 3.75$, there exists at least five co-existing equilibria of* (1) *with c the C^∞ but nowhere analytic kernel function given by* (15)*.*

Proof. The proof is obtained by running the script `script_proof_ex2_th2.m` which computes with interval arithmetic the coefficients of the radii polynomial $p(r)$ in (7). For each of the five numerically computed solution $\bar{x}$ in the MATLAB data file `ex2_coexisting_pts_mu3pt75.mat`, the code verifies the existence of an interval $\mathcal{I} = (r_{\min}, r_{\max})$ such that for each $r_0 \in \mathcal{I}$, $p(r_0) < 0$, and so by Theorem 1.5, there exists a unique $\tilde{a} \in B_{r_0}(\bar{a})$ such that $F(\tilde{a}) = 0$ with F given component-wise in (25). □

Theorem 3.5. *At $\mu = 4.5$, there exists at least five co-existing equilibria of the Fisher equation* (1) *with the kernel c given by* (15)*.*

Proof. The proof is obtained by running the script `script_proof_ex2_th3.m` which computes with interval arithmetic the coefficients of the radii polynomial $p(r)$ in (7). For each of the five numerically computed solution $\bar{x}$ in the MATLAB data file `ex2_coexisting_pts_mu4pt5.mat`, the code verifies the existence of an interval $\mathcal{I} = (r_{\min}, r_{\max})$ such that for each $r_0 \in \mathcal{I}$, $p(r_0) < 0$,

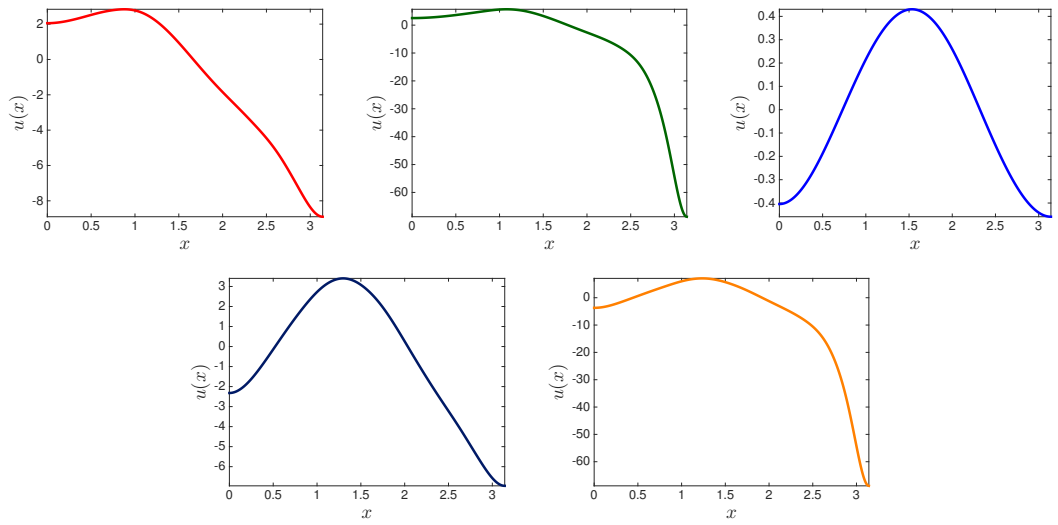

Figure 10: The five co-existing equilibrium solutions of Theorem 3.4. Each solution is a steady state of the Fisher equation at $\mu = 3.75$. The kernel function is the C^∞ but nowhere analytic kernel function given by (15).

and so by Theorem 1.5, there exists a unique $\tilde{a} \in B_{r_0}(\bar{a})$ such that $F(\tilde{a}) = 0$ with F given component-wise in (25). □

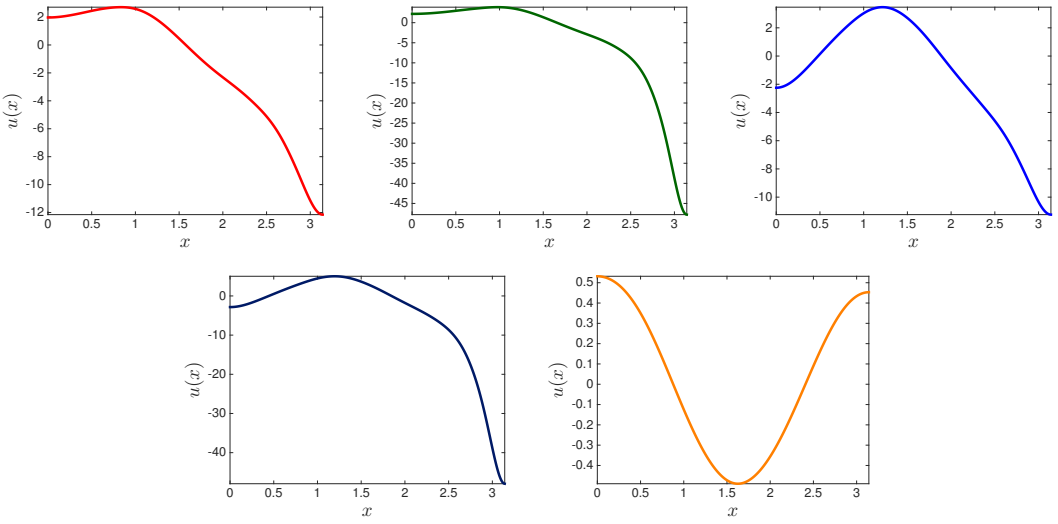

Figure 11: From Theorem 3.5, five co-existing equilibrium solutions of the Fisher equation at $\mu = 4.5$ with the C^∞ but nowhere analytic kernel function.

Theorem 3.6. *At $\mu = 1000$, there exists four equilibria of the Fisher equation* (1) *with $c(x)$ the kernel function given in* (15)*.*

Proof. The proof is obtained by running the script `script_proof_ex2_th4.m` which computes with interval arithmetic the coefficients of the radii polynomial $p(r)$ in (7). For each of the four numerically computed solution $\bar{x}$ in the MATLAB data file `ex2_coexisting_pts_mu1000.mat`, the code verifies the existence of an interval $\mathcal{I} = (r_{\min}, r_{\max})$ such that for each $r_0 \in \mathcal{I}$, $p(r_0) < 0$, and so by Theorem 1.5, there exists a unique $\tilde{a} \in B_{r_0}(\bar{a})$ such that $F(\tilde{a}) = 0$ with F given component-wise in (25). □

Solution's color	C^0-error	L^2-error	$\mathcal{I}$
red	1.4931e-06	3.2007e-07	[1.6844e-07 0.003135]
blue	1.3762e-06	2.95e-07	[1.5525e-07 0.0033917]
green	6.1271e-07	1.3135e-07	[6.9121e-08 0.0020115]
dark red	5.4091e-07	1.1596e-07	[6.1021e-08 0.0022134]

Figure 12: Different data for the proofs of Theorem 3.6 for the kernel function with Fourier coefficients given by (16). For each proof, $\mu = 1000$, $m = 500$, $q = 1.3$ and $M = 3000$.

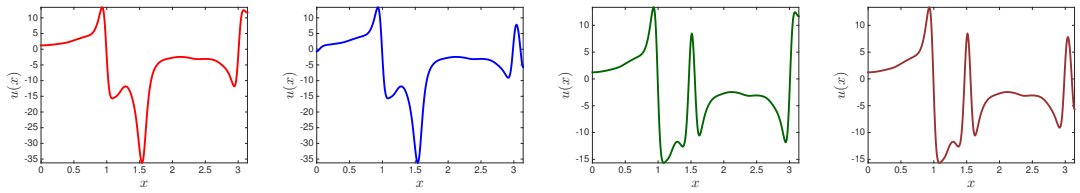

Figure 13: From Theorem 3.6, four co-existing equilibrium solutions of the Fisher equation at $\mu = 1000$ with the C^∞ but nowhere analytic kernel function.

3.2.3 The cubic B-spline kernel function

Consider the kernel C^2 piecewise cubic bump (B-spline) function as described in Section 1.3.3 with Fourier coefficients given by

$$c_n = \begin{cases} \dfrac{h^4}{\pi}, & n = 0 \\ \dfrac{4}{\pi n^4}\cos(nx_0)(1-\cos(nh))^2, & n \geq 1 \end{cases} \tag{53}$$

The function corresponds to a cubic spline *bump* of width $2h$ and of height 1. In this case,

$$|c_n| \leq \frac{16}{\pi n^4}, \qquad \text{for all } n \geq 1,$$

and so, the bound C_M satisfying is given by

$$C_M \stackrel{\text{def}}{=} \frac{16}{\pi M^4}, \qquad \text{for all } n \geq M.$$

If $n \geq M > 2N$ and $|n_1|, |n_2| \leq N$, then $n - n_1 - n_2 > 0$. In this case,

$$|c_{n-n_1-n_2}| \leq \frac{16}{\pi(n-n_1-n_2)^4} \leq \frac{16}{\pi h(n-2N)^4}.$$

Given $q \in (1,4]$, the function $x \mapsto \frac{x^q}{(x-2N)^4}$ is strictly decreasing for all $x > 2N$. Indeed, it has a derivative given by $h(x)\left((q-4)x - 2Nq\right)$ with $(q-4)x - 2Nq < 0 < h(x)$ for all $x > 2N$. Hence, for all $n \geq M$,

$$\begin{aligned}
|(c * \bar{a} * \bar{a})_n| &= \left| \sum_{n_1+n_2+n_3=n} \bar{a}_{n_1}\bar{a}_{n_2}c_{n_3} \right| \\
&\leq \sum_{n_1=-N}^{N} \sum_{n_2=-N}^{N} |\bar{a}_{n_1}||\bar{a}_{n_2}||c_{n-n_1-n_2}| \\
&\leq \left(\frac{16n^q}{\pi(n-2N)^4} \sum_{n_1=-N}^{N} \sum_{n_2=-N}^{N} |\bar{a}_{n_1}||\bar{a}_{n_2}| \right) \frac{1}{n^q} \\
&\leq \left(\frac{16M^q}{\pi(M-2N)^4} \right) \left(\sum_{j=-N}^{N} |\bar{a}_j| \right)^2 \frac{1}{\omega_n^q}.
\end{aligned}$$

Hence, recalling (37), set

$$\tilde{Y}_0^{(M)} \stackrel{\text{def}}{=} \left(\frac{\mu}{M^2-\mu} \right) \min \left\{ \alpha_M^{(3)} \|c\|_{\infty,q} \|\bar{a}\|_{\infty,q}^2 \ , \ \frac{16M^q}{\pi(M-2N)^4} \left(\sum_{j=-N}^{N} |\bar{a}_j| \right)^2 \right\}. \tag{54}$$

Combining the above value of C_M and the value of $\tilde{Y}_0^{(M)}$ given by (54) with the general bounds Y_0, Z_0, Z_1 and Z_2 given respectively by (38), (39), (45) and (47), we have all the bounds to define the radii polynomial $p(r)$ as defined in (7). The following result is proved using the radii polynomial approach.

Theorem 3.7. *For each point on Figure 14 the rigorous verification method was successful in proving the existence of a unique equilibria near the numerically computed solution.*

Proof. For each point in the MATLAB data files `ex3_cubic_spline_branchj.mat` with `j` $\in \{1,2,3,4,5\}$, the MATLAB script `script_proof_ex3_th1.m` computes the coefficients of the radii polynomial as defined in (7), with $M = 3000$ and $q = 1.3$. For each point, the code verifies the existence of an interval $\mathcal{I} = (r_{\min}, r_{\max})$ such that for each $r_0 \in \mathcal{I}$, $p(r_0) < 0$, and so by Theorem 1.5, there exists a unique $\tilde{a} \in B_{r_0}(\bar{a})$ such that $F(\tilde{a}) = 0$ with F given component-wise in (25). □

3.2.4 The piecewise linear spline kernel function

Consider the C^0 piecewise linear bump (spline) kernel function as described in Section 1.3.2 with Fourier coefficients given by

$$c_n = \begin{cases} \dfrac{Dh}{\pi}, & n = 0 \\ \dfrac{2D}{\pi h n^2} \cos(nx_0)\left(1 - \cos(nh)\right), & n \geq 1. \end{cases} \tag{55}$$

The function corresponds to a *bump* of width $2h$ and of height D. In this case,

$$|c_n| \leq \frac{4D}{\pi h n^2}, \qquad \text{for all } n \geq 1,$$

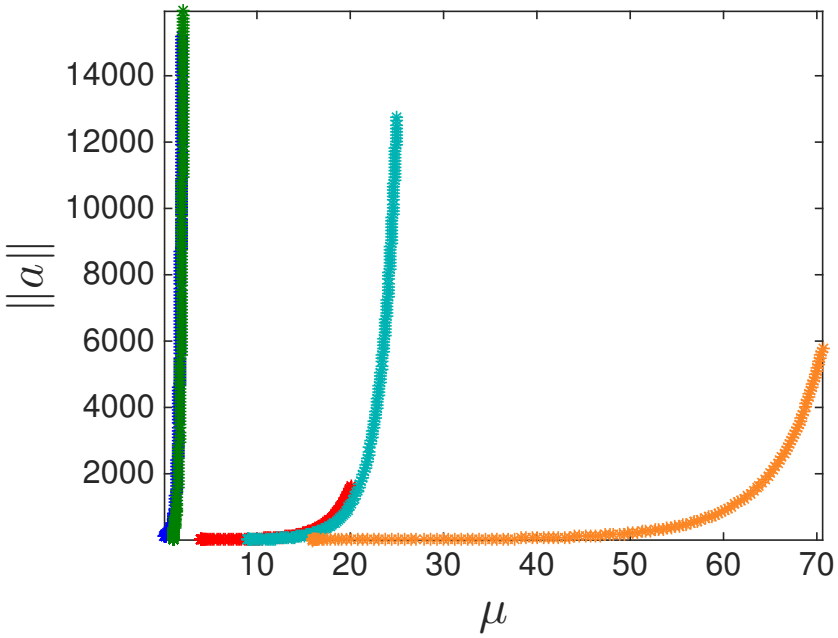

Figure 14: From Theorem 3.7, five branches of equilibria of the Fisher equation with the cubic B-spline kernel function with $h = 0.4$ and $x_0 = \pi/2$.

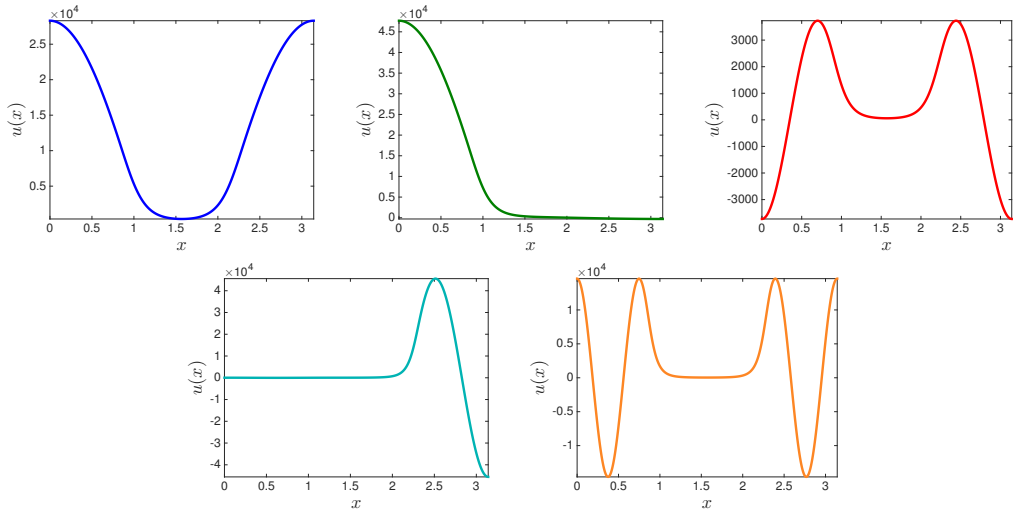

Figure 15: The most right points of each branch of Figure 14 as computed in Theorem 3.7.

and so, the bound C_M satisfying is given by

$$C_M \stackrel{\text{def}}{=} \frac{4D}{\pi h M^2}.$$

If $n \geq M > 2N$ and $|n_1|, |n_2| \leq N$, then $n - n_1 - n_2 > 0$. In this case,

$$|c_{n-n_1-n_2}| \leq \frac{4D}{\pi h(n-n_1-n_2)^2} \leq \frac{4D}{\pi h(n-2N)^2}.$$

Given $q \in (1,2]$, the function $x \mapsto \frac{x^q}{(x-2N)^2}$ is strictly decreasing for all $x > 2N$. Indeed, it has a derivative given by $h(x)\left((q-2)x - 2Nq\right)$ with $(q-2)x - 2Nq < 0 < h(x)$ for all $x > 2N$. Hence, for all $n \geq M$,

$$\begin{aligned}
|(c * \bar{a} * \bar{a})_n| &= \left| \sum_{n_1+n_2+n_3=n} \bar{a}_{n_1} \bar{a}_{n_2} c_{n_3} \right| \\
&\leq \sum_{n_1=-N}^{N} \sum_{n_2=-N}^{N} |\bar{a}_{n_1}||\bar{a}_{n_2}||c_{n-n_1-n_2}| \\
&\leq \left(\frac{4Dn^q}{\pi h(n-2N)^2} \sum_{n_1=-N}^{N} \sum_{n_2=-N}^{N} |\bar{a}_{n_1}||\bar{a}_{n_2}| \right) \frac{1}{n^q} \\
&\leq \left(\frac{4DM^q}{\pi h(M-2N)^2} \right) \left(\sum_{j=-N}^{N} |\bar{a}_j| \right)^2 \frac{1}{\omega_n^q}.
\end{aligned}$$

Hence, recalling (37), set

$$\tilde{Y}_0^{(M)} \stackrel{\text{def}}{=} \left(\frac{\mu}{M^2 - \mu} \right) \min \left\{ \alpha_M^{(3)} \|c\|_{\infty,q} \|\bar{a}\|_{\infty,q}^2 \, , \; \frac{4DM^q}{\pi h(M-2N)^2} \left(\sum_{j=-N}^{N} |\bar{a}_j| \right)^2 \right\}. \tag{56}$$

Combining the above value of C_M and the value of $\tilde{Y}_0^{(M)}$ given by (56) with the general bounds Y_0, Z_0, Z_1 and Z_2 given respectively by (38), (39), (45) and (47), we have all the bounds to define the radii polynomial $p(r)$ as defined in (7). The following result is proved using the radii polynomial approach.

Theorem 3.8. *For each point on Figure 16 the rigorous verification method was successful in proving the existence of a unique equilibria near the numerically computed solution.*

Proof. For each point in the MATLAB data files `ex4_linear_spline_branchj.mat` with `j` $\in \{1,2,3,4,5,6\}$, the MATLAB script `script_proof_ex4_th1.m` computes the coefficients of the radii polynomial as defined in (7), with $M = 3000$ and $q = 1.3$. For each point, the code verifies the existence of an interval $\mathcal{I} = (r_{\min}, r_{\max})$ such that for each $r_0 \in \mathcal{I}$, $p(r_0) < 0$, and so by Theorem 1.5, there exists a unique $\tilde{a} \in B_{r_0}(\bar{a})$ such that $F(\tilde{a}) = 0$ with F given component-wise in (25). □

3.2.5 The continuous kernel function which is not Lipschitz continuous

Consider a continuous kernel function which is not Lipschitz continuous as described in Section 1.3.1 with Fourier coefficients given by (8). In this case,

$$|c_n| \leq \frac{1}{n^{3/2}}, \qquad \text{for all } n \geq 1,$$

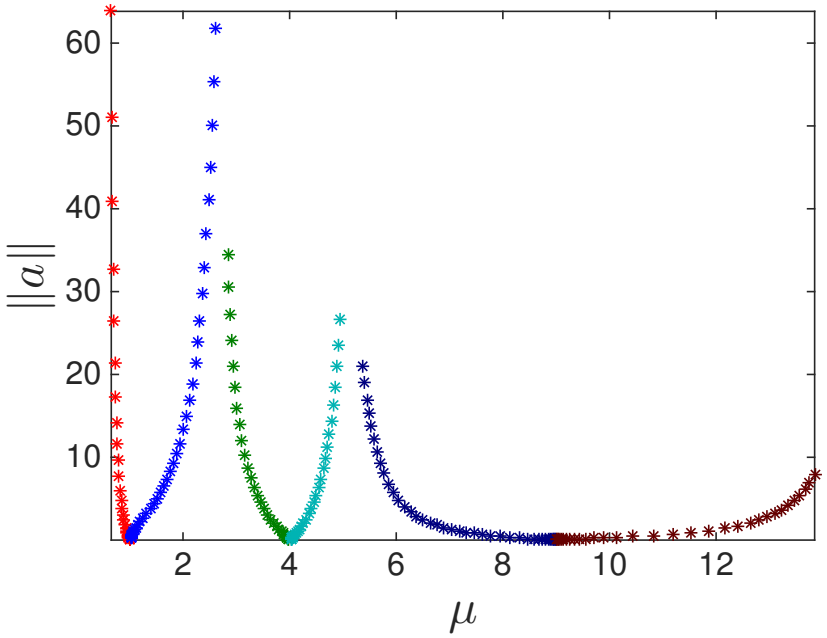

Figure 16: From Theorem 3.8, six branches of equilibria of the Fisher equation with the linear spline kernel function with $D = 1.5$, $h = 0.2$ and $x_0 = 1$.

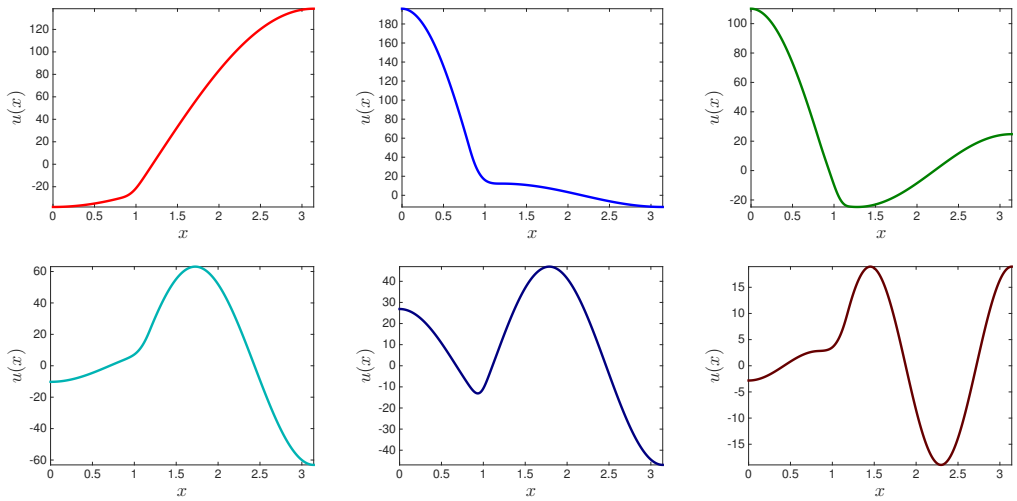

Figure 17: The top most points of each branch of Figure 16 as computed in Theorem 3.8.

and so, the bound C_M satisfying is given by

$$C_M \stackrel{\text{def}}{=} \frac{1}{M^{3/2}}.$$

If $|n_1|, |n_2| \leq N$, then $n - n_1 - n_2 > 0$. Hence, for all $n \geq M$,

$$\begin{aligned}
|(c * \bar{a} * \bar{a})_n| &= \left| \sum_{n_1+n_2+n_3=n} \bar{a}_{n_1} \bar{a}_{n_2} c_{n_3} \right| \\
&\leq \sum_{n_1=-N}^{N} \sum_{n_2=-N}^{N} |\bar{a}_{n_1}||\bar{a}_{n_2}||c_{n-n_1-n_2}| \\
&\leq \left(\frac{n^q}{\pi(n-2N)^{3/2}} \sum_{n_1=-N}^{N} \sum_{n_2=-N}^{N} |\bar{a}_{n_1}||\bar{a}_{n_2}| \right) \frac{1}{n^q} \\
&\leq \left(\frac{M^q}{\pi(M-2N)^{3/2}} \right) \left(\sum_{j=-N}^{N} |\bar{a}_j| \right)^2 \frac{1}{\omega_n^q}.
\end{aligned}$$

Hence, recalling (37), set

$$\tilde{Y}_0^{(M)} \stackrel{\text{def}}{=} \left(\frac{\mu}{M^2 - \mu} \right) \min \left\{ \alpha_M^{(3)} \|c\|_{\infty,q} \|\bar{a}\|_{\infty,q}^2 \ , \ \frac{M^q}{\pi(M-2N)^{3/2}} \left(\sum_{j=-N}^{N} |\bar{a}_j| \right)^2 \right\}. \tag{57}$$

Combining the above value of C_M and the value of $\tilde{Y}_0^{(M)}$ given by (57) with the general bounds Y_0, Z_0, Z_1 and Z_2 given respectively by (38), (39), (45) and (47), we have all the bounds to define the radii polynomial $p(r)$ as defined in (7). The following result is proved using the radii polynomial approach.

Theorem 3.9. *For each point on Figure 18 the rigorous verification method was successful in proving the existence of a unique equilibria near the numerically computed solution.*

Proof. For each point in the MATLAB data files `ex5_continuous_non_Lipschitz_branchj.mat` with `j` $\in \{1, 2, 3, 4, 5\}$, the MATLAB script `script_proof_ex5_th1.m` computes the coefficients of the radii polynomial as defined in (7), with $M = 3000$ and $q = 1.3$. For each point, the code verifies the existence of an interval $\mathcal{I} = (r_{\min}, r_{\max})$ such that for each $r_0 \in \mathcal{I}$, $p(r_0) < 0$, and so by Theorem 1.5, there exists a unique $\tilde{a} \in B_{r_0}(\bar{a})$ such that $F(\tilde{a}) = 0$ with F given component-wise in (25). □

3.3 Breakdown of analyticity

In this section we study a simple model of the breakdown of analyticity for a family of solutions of a differential equation. The idea is to study the spatially inhomogeneous term for Equation (1) as given by Equation (48) in Section 1.3.5, i.e. we take $c(x)$ the convolution of a piecewise linear bump function and a Poisson kernel.

This provides us with a one parameter family of analytic differential equations which, in the limit as $\sigma \to 1$, converges to a Lipschitz continuous equation. Then for each $\sigma < 1$ solutions of Equation (1) are analytic, yet the domain of analyticity is vanishing. The reduction of analyticity is reflected in the sequence space by the need to consider smaller and smaller exponential decay rates. So as σ increases we are forced to formulate the computer assisted proof in ℓ^1_ν with smaller and smaller $\nu > 1$. In other words, as $\sigma \to 1$ the analytic proof must get harder and harder.

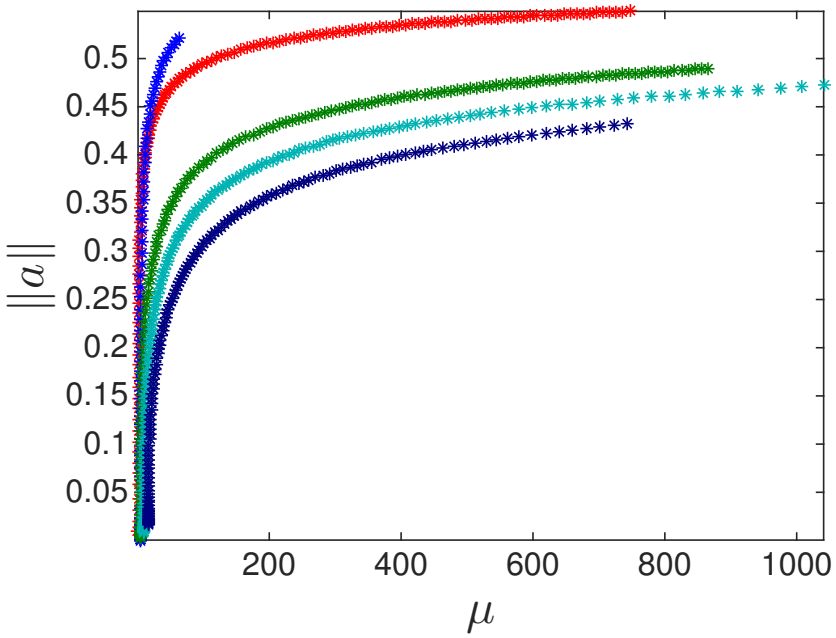

Figure 18: From Theorem 3.8, five branches of equilibria of the Fisher equation with the continuous kernel function which is not Lipschitz.

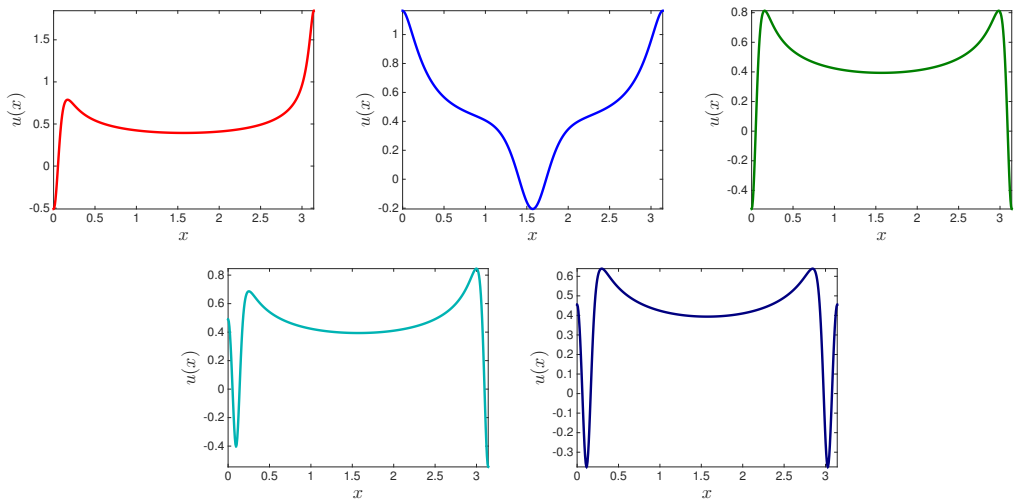

Figure 19: The top most points of each branch of Figure 18 as computed in Theorem 3.9.

If on the other hand we frame the computer assisted proof in a space of algebraic decay, then one decay rate can be used for all $0 \leq \sigma \leq 1$ (including $\sigma = 1$). This is clear when we consider explicitly the Fourier coefficients given by Equation (18), where the exponential decay rate is lost as $\sigma \to 1$ but the coefficients nevertheless satisfy an algebraic decay rate proportional to $1/n^2$ for all σ. In other words, in the space of algebraic decays the breakdown of analyticity is

σ	$\|F(\bar{a})\|_2$	$\mathcal{I}_{C^{\mathbf{k}}}$	$\mathcal{I}_{C^\omega}$
0.9	2.7775×10^{-15}	$[2.8568 \times 10^{-13}, 0.13218]$	$[7.2297 \times 10^{-13}, 0.21145]$
0.99	1.0832×10^{-14}	$[1.9542 \times 10^{-7}, 0.027686]$	$[7.2883 \times 10^{-8}, 0.061661]$
0.999	1.9709×10^{-14}	$[2.8492 \times 10^{-5}, 0.018852]$	$[0.00010876, 0.043469]$
0.9999	1.856×10^{-14}	$[0.00020865, 0.017749]$	$[0.0019178, 0.033383]$
0.99999	5.6312×10^{-14}	$[0.00021005, 0.017652]$	`failure`

Figure 20: Different data for the proofs with the analytic *bump* kernel function with Fourier coefficients given by (48). For all computations, we utilized 500 Fourier modes, and we fixed $D = 1.5$, $x_0 = 1$ and $h = 0.2$. For the proofs in the analytic category, we fixed $\nu = 1 + 10^{-12}$ and for the proofs in the $C^{\mathbf{k}}$ category, we used $q = 1.3$ and $M = 3000$.

not felt.

Figure 20 records the results of a series of rigorous numerical computations which provide concrete illustration of the phenomenon just described. The code performing the proofs is given by `script_breakdown.m` and is available at [43].

The results substantiate the claim that as $\sigma \to 1$ the analytic proof is getting more and more difficult (ultimately failing), while the C^k proof is very little effected. Note also that the numerical defect is small in all cases.

Remark 3.10. In the example above the mechanism governing the breakdown of analyticity is elementary, and it is not difficult to see exactly what terms are causing the proof to fail. Note that the Y_0 bound given in Section 2.1.1 (analytic category) contains the term

$$\mu \frac{\|c^\infty\| \|\bar{a}\|^2}{(N+1)^2 - \mu},$$

where N is the projection dimension. But in the ℓ^1_ν norm we see (by considering the geometric series) that

$$\|c^\infty\| \leq \frac{4D}{\pi h (N+1)^2} \frac{(\sigma\nu)^{N+2}}{1 - \sigma\nu}.$$

Moreover, $\|c^\infty\|_{\nu,1}$ approaches infinity as $\sigma \to 1$. Then for fixed N the proof will eventually break down as σ increases.

The interesting observation is that similar breakdown phenomenon could be detected (and overcome) using the same combination of computer assisted Fourier analysis in exponential and algebraic spaces, for more complicated problems where the mechanism of breakdown is less obvious. The discussion of this section shows that the algebraic decay spaces will be useful for computer assisted study of other problems involving breakdown of analyticity.

A Proof of Theorem 1.5

Define the operator $T\colon X \to X$ by

$$T(x) = x - AF(x).$$

The goal is to show that T is a contraction mapping the closed ball $\overline{B_{r_0}(\bar{x})}$ into itself, in which case the result follows from the contraction mapping theorem and the assumption that A is injective.

Observe that

$$DT(x) = I - ADF(x),$$

for all $x \in X$. Now, given $x \in \overline{B_{r_0}(\bar{x})}$ and applying the bounds (3), (4), (5), and (6), we obtain

$$\begin{aligned}\|DT(x)\|_{B(X)} &= \|I - ADF(x)\|_{B(X)} \\ &\leq \|I - AA^\dagger\|_{B(X)} + \|A[A^\dagger - DF(\bar{x})]\|_{B(X)} + \|A[DF(\bar{x}) - DF(x)]\|_{B(X)} \\ &\leq Z_0 + Z_1 + Z_2 r_0. \end{aligned} \tag{58}$$

We now show that T maps $\overline{B_{r_0}(\bar{x})}$ into itself (in fact into the interior $B_{r_0}(\bar{x})$). Let $x \in B_{r_0}(\bar{x})$ and apply the Mean Value Theorem to obtain

$$\begin{aligned}\|T(x) - \bar{x}\|_X &\leq \|T(x) - T(\bar{x})\|_X + \|T(\bar{x}) - \bar{x}\|_X \\ &\leq \sup_{b \in B_{r_0}(\bar{x})} \|DT(b)\|_{B(X)} \|x - \bar{x}\|_X + \|AF(\bar{x})\|_X \\ &\leq (Z_0 + Z_1 + Z_2 r_0) r_0 + Y_0\end{aligned}$$

where the last inequality follows from (58). Applying (7) and the assumption that $p(r_0) < 0$ implies that $\|T(x) - \bar{x}\|_X < r_0$, the desired result.

To see that T is a contraction on $B_{r_0}(\bar{x})$, let $a, b \in B_{r_0}(\bar{x})$ and apply (58) to obtain

$$\begin{aligned}\|T(a) - T(b)\|_X &\leq \sup_{b \in B_{r_0}(\bar{x})} \|DT(b)\|_{B(X)} \|a - b\|_X \\ &\leq (Z_0 + Z_1 + Z_2 r_0) \|a - b\|_X.\end{aligned}$$

Again, from the assumption that $p(r_0) < 0$, we it follows that

$$Z_0 + Z_1 + Z_2 r_0 + \frac{Y_0}{r_0} < 1.$$

Since $Y_0/r_0 \geq 0$ we conclude that $\kappa \stackrel{\text{def}}{=} Z_0 + Z_1 + Z_2 r_0 < 1$ and hence $T \colon B_{r_0}(\bar{x}) \to B_{r_0}(\bar{x})$ is a contraction with contraction constant $\kappa < 1$. Then there exists a unique fixed point of T in $\overline{B_{r_0}(\bar{x})}$. Since T maps $\overline{B_{r_0}(\bar{x})}$ into $B_{r_0}(\bar{x})$ the fixed point is bounded away from the boundary of the ball. Finally since A is injective, a fixed point of T is a zero of F (and vice versa), and the proof is complete.

B Convolution estimates in the $C^{\mathbf{k}}$ category

In this Appendix, we provide the necessary convolution estimates required to construct the radii polynomial. We decided to include all formulas and proofs so that the paper is self-contained. Note, however, that these analytic convolution estimates are taken directly from [31, 30] for estimates concerning quadratic and cubic nonlinearities.

First we define, for $k \geq 2$

$$\chi_k(q) = \left(\frac{q}{2-q} + \frac{q(q-1)}{2(3-q)} + \frac{q(q-1)}{2\left\lfloor \frac{k}{2} \right\rfloor} + \frac{2 - (2/3)^q}{\left\lfloor \frac{k}{2} \right\rfloor} - \frac{2 - (2/3)^q}{q-1} \right) \frac{1}{\left\lfloor \frac{k}{2} \right\rfloor^{q-1}}, \tag{59}$$

and $q^*(M)$ the unique zero of χ_M in $(1,2)$. Note that χ_M is increasing on $(1,2)$, goes to $-\infty$ as q goes to 1 and to ∞ as q goes to 2, so $q^*(M)$ is well defined.

Remark B.1. *$q^*(M)$, the unique zero of χ_M in $(1,2)$ defined in (59), is increasing in M and converges rather rapidly towards a bounded value. In particular, for $M \geq 100$ (which is always the case for the proofs presented in this work), one has that $q^*(M) \geq q^*(100) = 1.4730$. Numerically, the limit when M goes to ∞ is about 1.475.*

Define

$$\gamma_M^q = \gamma_M^q(\boldsymbol{K}) \stackrel{\text{def}}{=} \begin{cases} 2\sum_{k_1=1}^{\boldsymbol{K}} \frac{1}{k_1^q} + \frac{2}{(q-1)\boldsymbol{K}^{q-1}}, & \text{if } 1 < q < q^*(M) \\ 2\sum_{k_1=1}^{\boldsymbol{K}} \frac{1}{k_1^q} + \frac{2}{(q-1)\boldsymbol{K}^{q-1}} + 2\chi_M(q), & \text{if } q^*(M) \le q < 2 \\ 2\left(\frac{M}{M-1}\right)^q + \left(\frac{4\ln(M-2)}{M} + \frac{\pi^2-6}{3}\right)\left(\frac{2}{M}+\frac{1}{2}\right)^{q-2}, & \text{if } q \ge 2, \end{cases} \tag{60}$$

and let

$$\alpha_k^{(2)} = \alpha_k^{(2)}(q, M, \boldsymbol{K}) \stackrel{\text{def}}{=} \begin{cases} 1 + 2\sum_{k_1=1}^{\boldsymbol{K}} \frac{1}{k_1^{2q}} + \frac{2}{\boldsymbol{K}^{2q-1}(2q-1)}, & \text{if } k = 0 \\ \sum_{k_1=1}^{\boldsymbol{K}} \frac{2k^q}{k_1^q(k+k_1)^q} + \frac{2k^q}{(k+\boldsymbol{K}+1)^q\boldsymbol{K}^{q-1}(q-1)} & \\ \qquad +2 + \sum_{k_1=1}^{k-1} \frac{k^q}{k_1^q(k-k_1)^q}, & \text{if } 1 \le k \le M-1 \\ 2 + 2\sum_{k_1=1}^{\boldsymbol{K}} \frac{1}{k_1^q} + \frac{2}{\boldsymbol{K}^{q-1}(q-1)} + \gamma_M^q, & \text{if } k \ge M, \end{cases} \tag{61}$$

and for $k < 0$,

$$\alpha_k^{(2)} \stackrel{\text{def}}{=} \alpha_{|k|}^{(2)}.$$

Lemma B.2 (**Quadratic estimates**). *Fix a decay rate $q > 1$, and consider $\boldsymbol{K}$ and $M \ge 6$ computational parameters. Then, for any $k \in \mathbb{Z}$,*

$$\sum_{\substack{k_1+k_2=k \\ k_j \in \mathbb{Z}}} \frac{1}{\omega_{k_1}^q \omega_{k_2}^q} \le \frac{\alpha_k^{(2)}}{\omega_k^q}.$$

Proof. For $k = 0$,

$$\begin{aligned} \sum_{\substack{k_1+k_2=0 \\ k_j \in \mathbb{Z}}} \frac{1}{\omega_{k_1}^q \omega_{k_2}^q} &= 1 + 2\sum_{k_1=1}^{\boldsymbol{K}} \frac{1}{\omega_{k_1}^{2q}} + 2\sum_{k_1=\boldsymbol{K}+1}^{\infty} \frac{1}{\omega_{k_1}^{2q}} \le 1 + 2\sum_{k_1=1}^{\boldsymbol{K}} \frac{1}{\omega_{k_1}^{2q}} + \int_{\boldsymbol{K}}^{\infty} \frac{dx}{x^{2q}} \\ &\le 1 + 2\sum_{k_1=1}^{\boldsymbol{K}} \frac{1}{\omega_{k_1}^{2q}} + \frac{2}{\boldsymbol{K}^{2q-1}(2q-1)} = \frac{\alpha_0^{(2)}}{\omega_0^q}. \end{aligned}$$

For $1 \le k \le M-1$, and recalling that the one-dimensional weights (20),

$$\begin{aligned}
\sum_{\substack{k_1+k_2=k\\ k_j\in\mathbb{Z}}} \frac{1}{\omega_{k_1}^q \omega_{k_2}^q} &= \frac{1}{\omega_k^q}\left[\sum_{k_1=1}^{\boldsymbol{K}} \frac{2\omega_k^q}{\omega_{k_1}^q\omega_{k+k_1}^q} + \sum_{k_1=\boldsymbol{K}+1}^{\infty} \frac{2\omega_k^q}{\omega_{k_1}^q\omega_{k+k_1}^q} + \frac{2}{\omega_0^q} + \sum_{k_1=1}^{k-1} \frac{\omega_k^q}{\omega_{k_1}^q\omega_{k-k_1}^q}\right] \\
&\le \frac{1}{\omega_k^q}\left[\sum_{k_1=1}^{\boldsymbol{K}} \frac{2\omega_k^q}{\omega_{k_1}^q\omega_{k+k_1}^q} + \frac{2\omega_k^q}{(k+\boldsymbol{K}+1)^q}\int_{\boldsymbol{K}}^{\infty}\frac{dx}{x^q} + 2 + \sum_{k_1=1}^{k-1} \frac{\omega_k^q}{\omega_{k_1}^q\omega_{k-k_1}^q}\right] \\
&\le \frac{1}{\omega_k^q}\left[\sum_{k_1=1}^{\boldsymbol{K}} \frac{2\omega_k^q}{\omega_{k_1}^q\omega_{k+k_1}^q} + \frac{2\omega_k^q}{(k+\boldsymbol{K}+1)^q\boldsymbol{K}^{q-1}(q-1)} + 2 + \sum_{k_1=1}^{k-1} \frac{\omega_k^q}{\omega_{k_1}^q\omega_{k-k_1}^q}\right] = \frac{\alpha_k^{(2)}}{\omega_k^q}.
\end{aligned}$$

Finally, for $k \ge M$,

$$\begin{aligned}
\sum_{\substack{k_1+k_2=k\\ k_j\in\mathbb{Z}}} \frac{1}{\omega_{k_1}^q \omega_{k_2}^q} &= \frac{1}{\omega_k^q}\left[2\sum_{k_1=1}^{\infty} \frac{\omega_k^q}{\omega_{k_1}^q\omega_{k+k_1}^q} + \frac{2}{\omega_0^q} + \sum_{k_1=1}^{k-1} \frac{\omega_k^q}{\omega_{k_1}^q\omega_{k-k_1}^q}\right] \\
&\le \frac{1}{\omega_k^q}\left[2\sum_{k_1=1}^{\boldsymbol{K}} \frac{1}{\omega_{k_1}^q} + 2\sum_{k_1=\boldsymbol{K}+1}^{\infty} \frac{1}{\omega_{k_1}^q} + \frac{2}{\omega_0^q} + \gamma_M^q\right] \\
&\le \frac{1}{\omega_k^q}\left[2\sum_{k_1=1}^{\boldsymbol{K}} \frac{1}{\omega_{k_1}^q} + 2\int_{\boldsymbol{K}}^{\infty}\frac{dx}{x^q} + \frac{2}{\omega_0^q} + \gamma_M^q\right] \\
&\le \frac{1}{\omega_k^q}\left[\sum_{k_1=1}^{\boldsymbol{K}} \frac{2}{\omega_{k_1}^q} + \frac{2}{\boldsymbol{K}^{q-1}(q-1)} + 2 + \gamma_M^q\right] = \frac{\alpha_k^{(2)}}{\omega_k^q}. \quad \square
\end{aligned}$$

Lemma B.3 (Cubic estimates). *Given $q \ge 2$ and $M \ge 6$. Let*

$$\Sigma_a^* \stackrel{\text{def}}{=} \sum_{k_1=1}^{M-1} \frac{\alpha_{k_1}^{(2)} M^q}{k_1^q (M-k_1)^q} + \alpha_M^{(2)}\left(\gamma_M - \sum_{k_1=1}^{M-1}\frac{1}{k_1^q}\right),$$

$\tilde{\alpha}_M^{(2)} \stackrel{\text{def}}{=} \max\left\{\alpha_k^{(2)} \mid k = 0,\dots,M\right\}$, $\Sigma_b^* \stackrel{\text{def}}{=} \tilde{\alpha}_M^{(2)}\gamma_M$ *and* $\Sigma^* \stackrel{\text{def}}{=} \min\{\Sigma_a^*, \Sigma_b^*\}$. *Define the cubic asymptotic estimates $\alpha_k^{(3)} = \alpha_k^{(3)}(q,M)$ by*

$$\alpha_k^{(3)} \stackrel{\text{def}}{=} \begin{cases}
\alpha_0^{(2)} + 2\displaystyle\sum_{k_1=1}^{M-1}\frac{\alpha_{k_1}^{(2)}}{k_1^{2q}} + \frac{2\alpha_M^{(2)}}{(M-1)^{2q-1}(2q-1)}, & \text{if } k = 0 \\
\displaystyle\sum_{k_1=1}^{M-k}\frac{\alpha_{k+k_1}^{(2)}k^q}{k_1^q(k+k_1)^q} + \frac{\alpha_M^{(2)}k^q}{(M+1)^q(M-k)^{q-1}(q-1)} + \sum_{k_1=1}^{k-1}\frac{\alpha_{k_1}^{(2)}k^q}{k_1^q(k-k_1)^q} \\
\qquad + \displaystyle\sum_{k_1=1}^{M}\frac{\alpha_{k_1}^{(2)}k^q}{k_1^q(k+k_1)^q} + \frac{\alpha_M^{(2)}k^q}{(M+k+1)^q M^{q-1}(q-1)} + \alpha_k^{(2)} + \alpha_0^{(2)}, \\
\qquad\qquad \text{if } 1 \le k \le M-1 \\
\alpha_M^{(2)}\displaystyle\sum_{k_1=1}^{M}\frac{1}{k_1^q} + \frac{2\alpha_M^{(2)}}{M^{q-1}(q-1)} + \Sigma^* + \sum_{k_1=1}^{M}\frac{\alpha_{k_1}^{(2)}}{k_1^q} + \alpha_M^{(2)} + \alpha_0^{(2)}, & \text{if } k \ge M
\end{cases} \tag{62}$$

and for $k<0$,

$$\alpha_k^{(3)} \stackrel{\text{def}}{=} \alpha_{|k|}^{(3)}.$$

Then, for any $k\in\mathbb{Z}$,

$$\sum_{\substack{k_1+k_2+k_3=k\\ k_j\in\mathbb{Z}}} \frac{1}{\omega_{k_1}^q\omega_{k_2}^q\omega_{k_3}^q} \leq \frac{\alpha_k^{(3)}}{\omega_k^q}. \tag{63}$$

Proof. For $k=0$,

$$\sum_{\substack{k_1+k_2+k_3=0\\ k_j\in\mathbb{Z}}} \frac{1}{\omega_{k_1}^q\omega_{k_2}^q\omega_{k_3}^q} \quad\leq\quad \alpha_0^{(2)} + 2\sum_{k_1=1}^{M-1}\frac{\alpha_{k_1}^{(2)}}{\omega_{k_1}^{2q}} + \frac{2\alpha_M^{(2)}}{(M-1)^{2q-1}(2q-1)} = \frac{\alpha_0^{(3)}}{\omega_0^q}.$$

For $k>0$,

$$\begin{aligned}\sum_{\substack{k_1+k_2+k_3=k\\ k_j\in\mathbb{Z}}} \frac{1}{\omega_{k_1}^q\omega_{k_2}^q\omega_{k_3}^q} \quad\leq\quad & \sum_{k_1=1}^{\infty}\left[\frac{1}{\omega_{k_1}^q}\frac{\alpha_{k+k_1}^{(2)}}{\omega_{k+k_1}^q}\right] + \sum_{k_1=1}^{k-1}\left[\frac{1}{\omega_{k_1}^q}\frac{\alpha_{k-k_1}^{(2)}}{\omega_{k-k_1}^q}\right] + \sum_{k_1=1}^{\infty}\left[\frac{1}{\omega_{k+k_1}^q}\frac{\alpha_{k_1}^{(2)}}{\omega_{k_1}^q}\right]\\ & + \frac{1}{\omega_0^q}\frac{\alpha_k^{(2)}}{\omega_k^q} + \frac{1}{\omega_k^q}\frac{\alpha_0^{(2)}}{\omega_0^q}.\end{aligned}$$

For $k\in\{1,\ldots,M-1\}$, we have

$$\sum_{k_1=1}^{\infty}\frac{\alpha_{k+k_1}^{(2)}}{\omega_{k_1}^q\omega_{k+k_1}^q} \leq \frac{1}{\omega_k^q}\left[\sum_{k_1=1}^{M-k}\frac{\alpha_{k+k_1}^{(2)}\omega_k^q}{\omega_{k_1}^q\omega_{k+k_1}^q} + \frac{\alpha_M^{(2)}\omega_k^q}{(M+1)^q(M-k)^{q-1}(q-1)}\right].$$

Similarly,

$$\sum_{k_1=1}^{\infty}\frac{\alpha_{k_1}^{(2)}}{\omega_{k_1}^q\omega_{k+k_1}^q} \leq \frac{1}{\omega_k^q}\left[\sum_{k_1=1}^{M}\frac{\alpha_{k_1}^{(2)}\omega_k^q}{\omega_{k_1}^q\omega_{k+k_1}^q} + \frac{\alpha_M^{(2)}\omega_k^q}{(M+k+1)^q M^{q-1}(q-1)}\right].$$

From the definition of $\alpha_k^{(3)}$ for $k\in\{1,\ldots,M-1\}$, one gets that

$$\sum_{\substack{k_1+k_2+k_3=k\\ k_j\in\mathbb{Z}}} \frac{1}{\omega_{k_1}^q\omega_{k_2}^q\omega_{k_3}^q} \leq \frac{\alpha_k^{(3)}}{\omega_k^q}.$$

For $k\geq M$, one gets that

$$\sum_{k_1=1}^{\infty}\frac{\alpha_{k+k_1}^{(2)}}{\omega_{k_1}^q\omega_{k+k_1}^q} \leq \frac{1}{\omega_k^q}\left[\alpha_M^{(2)}\sum_{k_1=1}^{M}\frac{1}{\omega_{k_1}^q} + \frac{\alpha_M^{(2)}}{M^{q-1}(q-1)}\right].$$

Moreover,

$$\begin{aligned}
\sum_{k_1=1}^{k-1} \frac{\alpha_{k_1}^{(2)}}{\omega_{k_1}^q \omega_{k-k_1}^q} &= \sum_{k_1=1}^{M-1} \frac{\alpha_{k_1}^{(2)}}{\omega_{k_1}^q \omega_{k-k_1}^q} + \frac{1}{\omega_k^q} \sum_{k_1=M}^{k-1} \frac{\omega_k^q \alpha_{k_1}^{(2)}}{\omega_{k_1}^q \omega_{k-k_1}^q} \\
&\leq \frac{1}{\omega_k^q} \sum_{k_1=1}^{M-1} \frac{\alpha_{k_1}^{(2)}}{\omega_{k_1}^q \left(1-\frac{k_1}{k}\right)^q} + \frac{\alpha_M^{(2)}}{\omega_k^q} \sum_{k_1=M}^{k-1} \frac{\omega_k^q}{\omega_{k_1}^q \omega_{k-k_1}^q} \\
&\leq \frac{1}{\omega_k^q} \left[\sum_{k_1=1}^{M-1} \frac{\alpha_{k_1}^{(2)}}{\omega_{k_1}^q \left(1-\frac{k_1}{M}\right)^q} + \alpha_M^{(2)} \left(\sum_{k_1=1}^{k-1} \frac{\omega_k^q}{\omega_{k_1}^q \omega_{k-k_1}^q} - \sum_{k_1=1}^{M-1} \frac{\omega_k^q}{\omega_{k_1}^q \omega_{k-k_1}^q} \right) \right] \\
&\leq \frac{1}{\omega_k^q} \left[\sum_{k_1=1}^{M-1} \frac{\alpha_{k_1}^{(2)} M^q}{\omega_{k_1}^q \left(M-k_1\right)^q} + \alpha_M^{(2)} \left(\gamma_M - \sum_{k_1=1}^{M-1} \frac{1}{\omega_{k_1}^q} \right) \right] = \frac{1}{\omega_k^q} \Sigma_a^*.
\end{aligned}$$

Hence,

$$\sum_{k_1=1}^{k-1} \frac{\alpha_{k_1}^{(2)}}{\omega_{k_1}^q \omega_{k-k_1}^q} \leq \frac{\tilde{\alpha}_M^{(2)}}{\omega_k^q} \gamma_M = \frac{1}{\omega_k^q} \Sigma_b^*.$$

Recalling that $\Sigma^* = \min\{\Sigma_a^*, \Sigma_b^*\}$, one gets that $\displaystyle\sum_{k_1=1}^{k-1} \frac{\alpha_{k_1}^{(2)}}{\omega_{k_1}^q \omega_{k-k_1}^q} \leq \frac{1}{\omega_k^q} \Sigma^*$. Also,

$$\sum_{k_1=1}^{\infty} \frac{\alpha_{k_1}^{(2)}}{\omega_{k_1}^q \omega_{k+k_1}^q} \leq \frac{1}{\omega_k^q} \left[\sum_{k_1=1}^{M} \frac{\alpha_{k_1}^{(2)}}{\omega_{k_1}^q} + \frac{\alpha_M^{(2)}}{M^{q-1}(q-1)} \right].$$

Combining the above inequalities, we get, for the case $k \geq M$,

$$\begin{aligned}
\sum_{\substack{k_1+k_2+k_3=k \\ k_j \in \mathbb{Z}}} \frac{1}{\omega_{k_1}^q \omega_{k_2}^q \omega_{k_3}^q} \leq \frac{1}{\omega_k^q} \Bigg[& \alpha_M^{(2)} \sum_{k_1=1}^{M} \frac{1}{\omega_{k_1}^q} + \frac{2\alpha_M^{(2)}}{M^{q-1}(q-1)} + \Sigma^* \\
& + \sum_{k_1=1}^{M} \frac{\alpha_{k_1}^{(2)}}{\omega_{k_1}^q} + \alpha_M^{(2)} + \alpha_0^{(2)} \Bigg] = \frac{\alpha_k^{(3)}}{\omega_k^q}. \quad \square
\end{aligned}$$

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