The radii polynomials for the Ohta-Kawasaki equation

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The Ohta-Kawasaki equation

$$\begin{cases} u_t = -u_{xxxx} - (\psi_\lambda(u))_{xx} - \lambda\sigma u, & \text{for } x \in [0,\pi], \\ u_x(t,0) = u_x(t,\pi) = 0, \\ u_{xxx}(t,0) = u_{xxx}(t,\pi) = 0, \\ \int_0^\pi u(0,x)dx = 0. \end{cases} \tag{0.1}$$

We begin with some background.

1 Elementary functional analytic background

The space used in the present example is

$$\ell^1_\nu \stackrel{\text{def}}{=} \left\{ c = \{c_k\}_{k\geq 1} \,:\, \|c\|_{1,\nu} < \infty \right\}, \tag{1.1}$$

equipped with a "weighted ell one norm"

$$\|c\|_{1,\nu} \stackrel{\text{def}}{=} 2\sum_{k\geq 1} |c_k|\nu^k = \sum_{k\in\mathbb{Z}\setminus\{0\}} |c_{|k|}|\nu^{|k|}, \tag{1.2}$$

for some fixed weight $\nu \geq 1$.

We note that ℓ^1_ν is a Banach space and moreover has the property of being a Banach algebra under discrete convolution defined as

$$a * b = \left\{ \sum_{\substack{k_1,k_2\in\mathbb{Z}\setminus\{0\} \\ k_1+k_2=k}} a_{|k_1|}b_{|k_2|} \right\}_{k\geq 1}, \qquad a = \{a_k\}_{k\geq 1}, b = \{b_k\}_{k\geq 1} \in \ell^1_\nu.$$

More explicitly, we have the following.

Lemma 1. *If $\nu \geq 1$ and $a, b \in \ell^1_\nu$, then $a * b \in \ell^1_\nu$ and*

$$\|a * b\|_{1,\nu} \leq \|a\|_{1,\nu}\|b\|_{1,\nu}. \tag{1.3}$$

The estimates are most transparent when viewed in the context of the Banach space dual of ℓ^1_ν. For a sequence of real numbers $c = \{c_n\}_{n=1}^\infty$ define the ν-weighted supremum norm

$$\|c\|_{\infty,\nu^{-1}} \stackrel{\text{def}}{=} \frac{1}{2}\sup_{n\geq 1} \frac{|c_n|}{\nu^n}, \tag{1.4}$$

and let

$$\ell_\nu^\infty \stackrel{\text{def}}{=} \left\{c = \{c_n\}_{n=1}^\infty : \|c\|_{\infty,\nu^{-1}} < \infty\right\}.$$

From the definition of the norm in (1.4), it follows that given $c \in \ell_\nu^\infty$,

$$|c_k| \le 2\nu^k \|c\|_{\infty,\nu^{-1}}, \quad \forall\ k \ge 1. \tag{1.5}$$

Then, given $a \in \ell_\nu^1$ and $c \in \ell_\nu^\infty$, we conclude by (1.5) that

$$\left|\sum_{k\ge1} c_k a_k\right| \le \sum_{k\ge1} |c_k||a_k| \le \|c\|_{\infty,\nu^{-1}} \left(2\sum_{k\ge1} |a_k|\nu^k\right) = \|c\|_{\infty,\nu^{-1}}\|a\|_{1,\nu}. \tag{1.6}$$

This bound is used to estimate linear operators of the following type. Denote by $B(\ell_\nu^1, \ell_\nu^1)$ the space of bounded linear operators from ℓ_ν^1 to ℓ_ν^1 and by $\|\cdot\|_{B(\ell_\nu^1,\ell_\nu^1)}$ the operator norm.

Corollary 1. *Let $A^{(m)}$ be an $(m-1)\times(m-1)$ matrix, $\{\mu_n\}_{n=m}^\infty$ be a sequence of numbers with*

$$|\mu_n| \le |\mu_m|,$$

for all $n \ge m$, and $A\colon \ell_\nu^1 \to \ell_\nu^1$ be the linear operator defined by

$$A(a) = \begin{pmatrix} A^{(m)} & & 0 & \\ & \mu_m & & \\ 0 & & \mu_{m+1} & \\ & & & \ddots \end{pmatrix} \begin{bmatrix} a^{(m)} \\ a_m \\ a_{m+1} \\ \vdots \end{bmatrix}.$$

Here $a^{(m)} = (a_1, \dots, a_{m-1})^T \in \mathbb{R}^{m-1}$. Then $A \in B(\ell_\nu^1, \ell_\nu^1)$ is a bounded linear operator and

$$\|A\|_{B(\ell_\nu^1,\ell_\nu^1)} = \frac{1}{2}\max(K, |\mu_m|), \tag{1.7}$$

where

$$K \stackrel{\text{def}}{=} \max_{1\le n\le m-1} \frac{1}{\nu^n} \sum_{k=1}^{m-1} |A_{k,n}|\nu^k.$$

Proof. We have that

$$\begin{aligned}
\|A\|_{B(\ell_\nu^1,\ell_\nu^1)} &= \sup_{\|a\|_\nu=1} \|Aa\|_{1,\nu} \\
&= \sup_{\|a\|_\nu=1} \left(2\sum_{n=1}^{m-1}\left|\sum_{k=1}^{m-1} A_{n,k}a_k\right|\nu^n + 2\sum_{n=m}^\infty |\mu_n a_n|\nu^n\right) \\
&\le \sup_{\|a\|_\nu=1} \left(2\sum_{n=1}^{m-1}\left(\sum_{k=1}^{m-1}|A_{k,n}|\nu^k\right)|a_n| + 2\sum_{n=m}^\infty |\mu_n\nu^n||a_n|\right) \\
&= \sup_{\|a\|_\nu=1} \sum_{n=1}^\infty |a_n||c_n|,
\end{aligned}$$

where

$$c_n \stackrel{\text{def}}{=} \begin{cases} 2\displaystyle\sum_{k=1}^{m-1}|A_{k,n}|\nu^k, & \text{if } 1\le n\le m-1 \\ 2|\mu_n|\nu^n, & \text{if } n \ge m. \end{cases}$$

Note that $c = \{c_n\}_{n=1}^{\infty} \in \ell_\nu^\infty$, since

$$\|c\|_\nu^\infty = \frac{1}{2} \sup_{n\geq 1} \frac{|c_n|}{\nu^n} = \max(K, |\mu_m|),$$

with K and μ_m as given in the hypothesis of the corollary. We obtain the desired bound on $\|A\|_{B(\ell_\nu^1, \ell_\nu^1)}$ by applying (1.6). $\square$

Remarks 1. *Note that*

$$\sup_{\|v\|_{1,\nu}\leq 1} |(b * \widehat{v})_k| \leq \sup_{k'\geq m} \frac{|b_{|k-k'|} + b_{|k+k'|}|}{2\nu^{k'}} \stackrel{\text{def}}{=} \hat{\mathcal{Q}}_k(b). \tag{1.8}$$

which is useful when computing the Z_1 bound.

2 The set-up of the zero finding problem

Plugging the cosine Fourier expansion

$$u(x) = \sum_{k\in\mathbb{Z}} a_k e^{ikx}, \quad (a_k \in \mathbb{R}, \quad a_{-k} = a_k, \quad a_0 = 0)$$

in the steady states equation

$$-u_{xxxx} - (\psi_\lambda(u))_{xx} - \lambda\sigma u = 0,$$

we obtain

$$f_k \stackrel{\text{def}}{=} \mu_k a_k - \lambda k^2 (a^3)_k = 0, \tag{2.1}$$

where

$$\mu_k = \mu_k(\lambda, \sigma) \stackrel{\text{def}}{=} -k^4 + \lambda k^2 - \lambda\sigma, \tag{2.2}$$

where $a = (a_k)_{k\geq 1}$, and where

$$(a^3)_k \stackrel{\text{def}}{=} \sum_{\substack{k_1+k_2+k_3=k \\ k_i\in\mathbb{Z}\setminus\{0\}}} a_{|k_1|} a_{|k_2|} a_{|k_3|}.$$

The relations $f_k \in \mathbb{R}$, $g_{-k} = g_k$ and $g_0 = 0$ implies that we only need to solve $f_k = 0$ for $k \geq 1$. Set $f \stackrel{\text{def}}{=} (f_k)_{k\geq 1}$. The linear operators $A^\dagger$ and A are defined here as

$$(A^\dagger h)_k = \begin{cases} \left(Df^{(m)}(\bar{a}) h^{(m)}\right)_k & \text{for } 1 \leq k < m \\ \mu_k h_k & \text{for } k \geq m \end{cases}$$

and

$$(Ah)_k = \begin{cases} \left(A^{(m)} h^{(m)}\right)_k & \text{for } 1 \leq k < m \\ \frac{1}{\mu_k} h_k & \text{for } k \geq m. \end{cases}$$

3 The bounds

3.1 The Y_0 bound

Observe first that the nonlinear term of $f(\bar{a})$ involves the convolution product $(\bar{a} * \bar{a} * \bar{a})_k$, which vanishes for $k \geq 3m - 2$. This implies that $(f(\bar{a}))_k = 0$ for all $k \geq 3m - 2$. We set

$$Y_0 \stackrel{\text{def}}{=} 2 \sum_{k=1}^{m-1} \left| \left(A^{(m)} f^{(m)}(\bar{a})\right)_k \right| \nu^k + 2 \sum_{k=m}^{3m-3} \left| \frac{1}{\mu_k} f_k(\bar{a}) \right| \nu^k. \tag{3.1}$$

3.2 The Z_0 bound

Let $B \overset{\text{def}}{=} I - AA^\dagger$, we remark that $B_{n_1,n_2} = 0$ whenever $n_1 \geq m$ or $n_2 \geq m$.

$$\|B\|_{B(\ell^1_\nu)} = \max_{j=1,\ldots,m-1} \frac{1}{\nu^j} \sum_{i=1}^{m-1} |B_{i,j}|\nu^i = Z_0. \tag{3.2}$$

3.3 The Z_1 bound

Recall that we look for the bound $\|A[DF(\bar{a}) - A^\dagger]\|_{B(\ell^1_\nu)} \leq Z_1$. Given $h \in \ell^1_\nu$ with $\|h\|_{1,\nu} \leq 1$, set

$$z \overset{\text{def}}{=} [DF(\bar{a}) - A^\dagger]h.$$

Since in z some of the terms involving $(h_k)_{k=0}^{m-1}$ will cancel, it is useful to introduce $\widehat{h}$ as follows:

$$\widehat{h}_k \overset{\text{def}}{=} \begin{cases} 0 & \text{if } 1 \leq k < m, \\ h_k & \text{if } k \geq m. \end{cases}$$

Then,

$$z_k = \begin{cases} -3\lambda k^2 (\bar{a} * \bar{a} * \widehat{h})_k, & \text{for } k = 1, \ldots, m-1 \\ 3\lambda k^2 (\bar{a} * \bar{a} * h)_k & \text{for } k \geq m \end{cases}$$

By (1.8), we get that

$$|z_k| \leq 3|\lambda| k^2 \hat{\mathcal{Q}}_k(\bar{a} * \bar{a}), \qquad \text{for all } k = 1, \ldots, m-1.$$

Hence,

$$\begin{aligned}
\|Az\|_{1,\nu} &\leq \sum_{j=1}^{2} \|A_{1,j} z_j\|_{1,\nu} \\
&= \sum_{k=0}^{m-1} \left| \left(A_{1,1}^{(m)} z_1^{(m)}\right)_k \right| \nu^k + \sum_{k \geq m} \frac{1}{k} |(z_2)_k| \nu^k \\
&\leq 3|\lambda| \sum_{k=0}^{m-1} \left| \left(|A_{1,1}^{(m)}| \hat{\mathcal{Q}}^{(m)}(\bar{a} * \bar{a})\right)_k \right| \nu^k + \frac{1}{2m} \left(2 \sum_{k \geq m} |(z_2)_k| \nu^k \right) \\
&\leq 3|\lambda| \sum_{k=0}^{m-1} \left| \left(|A_{1,1}^{(m)}| \hat{\mathcal{Q}}^{(m)}(\bar{a} * \bar{a})\right)_k \right| \nu^k + \frac{1}{2m} \|h_2\|_{1,\nu} \\
&\leq 3|\lambda| \sum_{k=0}^{m-1} \left| \left(|A_{1,1}^{(m)}| \hat{\mathcal{Q}}^{(m)}(\bar{a} * \bar{a})\right)_k \right| \nu^k + \frac{1}{2m} \overset{\text{def}}{=} Z_1^{(1)},
\end{aligned}$$

and similarly, now also using the Banach algebra property, then

$$\begin{aligned}
\|(Az)_2\|_{1,\nu} &\le \sum_{j=1}^{2} \|A_{2,j} z_j\|_{1,\nu} \\
&= \|A_{2,1} z_1\|_{1,\nu} \\
&= \sum_{k=0}^{m-1} \left|\left(A_{2,1}^{(m)} z_1^{(m)}\right)_k\right| \nu^k + \sum_{k \ge m} \frac{1}{k} |(z_1)_k| \nu^k \\
&\le 3|\lambda| \sum_{k=0}^{m-1} \left|\left(|A_{2,1}^{(m)}| \hat{\mathcal{Q}}^{(m)} (\bar{a} * \bar{a})\right)_k\right| \nu^k + \frac{1}{2m} \left(2 \sum_{k \ge m} |(z_1)_k| \nu^k \right) \\
&\le 3|\lambda| \sum_{k=0}^{m-1} \left|\left(|A_{2,1}^{(m)}| \hat{\mathcal{Q}}^{(m)} (\bar{a} * \bar{a})\right)_k\right| \nu^k + \frac{|\lambda|}{2m} \left(\|h_1\|_{1,\nu} + 3(\|\bar{a}\|_{1,\nu})^2 \|h_1\|_{1,\nu}\right) \\
&\le 3|\lambda| \sum_{k=0}^{m-1} \left|\left(|A_{2,1}^{(m)}| \hat{\mathcal{Q}}^{(m)} (\bar{a} * \bar{a})\right)_k\right| \nu^k + \frac{|\lambda|}{2m} \left(1 + 3(\|\bar{a}\|_{1,\nu})^2\right) \stackrel{\text{def}}{=} Z_1^{(2)},
\end{aligned}$$

We thus define

$$Z_1 \stackrel{\text{def}}{=} \max\left(Z_1^{(1)}, Z_1^{(2)}\right). \tag{3.3}$$

3.4 The Z_2 bound

Recall that

$$\begin{pmatrix} (F_1(x))_k \\ (F_2(x))_k \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} -kb_k + \lambda\left(a_k - (a^3)_k\right) \\ -ka_k + b_k \end{pmatrix} = 0.$$

Let $c = (c_1, c_2) \in B_r(\bar{x})$, that is $\|c - \bar{x}\|_X = \max(\|c_1 - \bar{a}\|_{1,\nu}, \|c_2 - \bar{b}\|_{1,\nu}) \le r$. Given $\|h\|_X \le 1$, note that $([DF_2(c) - DF_2(\bar{x})]h)_k = 0$ and that

$$([DF_1(c) - DF_1(\bar{x})]h)_k = -3\lambda\left((c_1 * c_1 - \bar{a} * \bar{a}) * h_1\right)_k$$

so that

$$\begin{aligned}
\|A[DF(c) - DF(\bar{x})]\|_{B(X)} &= \sup_{\|h\|_X \le 1} \|A[DF(c) - DF(\bar{x})]h\|_X \\
&\le \|A\|_{B(X)} \sup_{\|h\|_X \le 1} \|[DF(c) - DF(\bar{x})]h\|_X \\
&= 3|\lambda| \|A\|_{B(X)} \sup_{\|h\|_X \le 1} \|(c_1 - \bar{a}) * (c_1 + \bar{a}) * h_1\|_{1,\nu} \\
&\le 3|\lambda| \|A\|_{B(X)} \sup_{\|h\|_X \le 1} \|c_1 - \bar{a}\|_{1,\nu} \|c_1 + \bar{a}\|_{1,\nu} \|h_1\|_{1,\nu} \\
&\le 3|\lambda| \|A\|_{B(X)} r(\|c_1\|_{1,\nu} + \|\bar{a}\|_{1,\nu}) \\
&\le 3|\lambda| \|A\|_{B(X)} r(r + 2\|\bar{a}\|_{1,\nu}).
\end{aligned}$$

Then, assuming a loose a priori bound $r \le 1$ on the radius, we set

$$Z_2 \stackrel{\text{def}}{=} 3|\lambda| \|A\|_{B(X)} (1 + 2\|\bar{a}\|_{1,\nu}), \tag{3.4}$$

where

$$\|A\|_{B(X)} = \max\left(\|A_{1,1}\|_{B(\ell_\nu^1)} + \|A_{1,2}\|_{B(\ell_\nu^1)}, \|A_{2,1}\|_{B(\ell_\nu^1)} + \|A_{2,2}\|_{B(\ell_\nu^1)}\right).$$