Estimates for Allen-Cahn problem in 2D

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1 The problem

We consider the equation

$$0 = \Delta u + \lambda(u - u^3)$$

with Neumann boundary conditions on the square $[0, \pi] \times [0, \pi]$.

We perform the cosine transform

$$u = \sum_{k \in \mathbb{Z}^2} a_k e^{ik \cdot x} = \sum_{k \in \mathbb{N}^2} m_k a_k \cos(k_1 x_1) \cos(k_2 x_2)$$

where the multiplicities are

$$m_k = m_{k_1,k_2} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{for } k_1 = k_2 = 0 \\ 2 & \text{for } k_1 = 0, k_2 > 0 \\ 2 & \text{for } k_1 > 0, k_2 = 0 \\ 4 & \text{for } k_1 > 0, k_2 > 0. \end{cases}$$

We will from now on always assume $a_{k_1,k_2} = a_{|k_1|,|k_2|} \in \mathbb{R}$. We write for $k \in \mathbb{N}^2$

$$\mathbf{k}^2 \stackrel{\text{def}}{=} k_1^2 + k_2^2$$
$$|k| \stackrel{\text{def}}{=} \max\{k_1, k_2\}.$$

The equations for the unknowns $(a_k)_{k \in \mathbb{N}^2}$ become

$$f_k \stackrel{\text{def}}{=} m_k[(-\mathbf{k}^2 + \lambda)a_k - \lambda(a * a * a)_k], \tag{1}$$

with the usual convolution.

2 Sequence space

The norm in Fourier space will be an (exponentially) weighted ℓ^1-norm:

$$\|a\|_\nu \stackrel{\text{def}}{=} \sum_{k \in \mathbb{N}^2} m_k |c_k| \, \nu^{|k|} = \sum_{k \in \mathbb{N}^2} |c_k| \omega_k,$$

with $\omega_k \stackrel{\text{def}}{=} m_k \nu^{|k|}$ and $\nu \geq 1$ (one may alternatively use another norm on k in the exponent of ν, for example $k_1 + k_2$). One nice thing about the norm is that $\|a * b\|_\nu \leq \|a\|_\nu \|b\|_\nu$. This makes our space

$$X = \{(a_k)_{k \in \mathbb{N}^2} \, : \, a_k \in \mathbb{R} \, , \, \|a\|_\nu < \infty\} \tag{2}$$

into a Banach algebra. The operator norm on X will be denoted by

$$\|B\|_\omega = \sup_{k \in \mathbb{N}^2} \frac{1}{\omega_k} \sum_{n \in \mathbb{N}^2} |B_{nk}| \omega_n,$$

and we will abuse notation slightly by also using it for finite matrices.

3 General setup

We will compute in the finite dimensional supspace (for some largish $K \in \mathbb{N}$)

$$X_F = \{a \in X \,:\, a_k = 0 \text{ for all } |k| \geq K+1\},$$

with corresponding projections

$$\begin{aligned} a_F^0 &= \pi_{X_F} a \\ a_\infty^0 &= a - a_F. \end{aligned}$$

We will write $a = (a_F, a_\infty)$, where $a_F \in \mathbb{R}^{(K+1)^2}$. We find a numerical approximation $\bar{a} \in X_F$ such that $f_F(\bar{a}) \approx 0$.

With f defined in (1) we now introduce the fixed point operator in X as

$$T(a) \stackrel{\text{def}}{=} a - Af(a).$$

Here A is a linear (block-diagonal) operator of the form

$$\begin{aligned} (Aa)_F &= A_F a_F \\ (Aa)_k &= \frac{1}{\mu_k} a_k \qquad \text{for } |k| > K. \end{aligned}$$

The matrix A_F is determined via a computer calculation, namely a numerical (i.e. not exact) inverse of the Jacobian J_F of the finite dimensional map $a_F \to f_F(a_F^0))$. Furthermore,

$$\mu_k \stackrel{\text{def}}{=} m_k[-\mathbf{k}^2 + \lambda].$$

We denote by B the unit ball in X:

$$B = \{(v_k)_{k\in\mathbb{N}^2} \,:\, \|v\|_\nu \leq 1\}.$$

We need Y and $Z(r)$ such that

$$\begin{aligned} \|T(\bar{a}) - \bar{a}\|_\nu &\leq Y, \\ \sup_{w,v\in B} \|DT(\bar{a} + rw)rv\|_\nu &\leq Z(r). \end{aligned}$$

The contraction mapping theorem then ensures that if

$$Y + Z(\tilde{r}) < \tilde{r} \qquad \text{for some } \tilde{r} > 0,$$

then we have a unique (genuine) solution in a ball of radius $\tilde{r}$ around the numerical guess $\bar{a}$.

4 bound Y

All nonvanishing terms in $F(\bar{a})$ can just be evaluated explicitely using interval arithmetic. Putting it all together we obtain,

$$Y = \|A_F f_F(\bar{a})\|_\nu + \sum_{K<|k|\leq 3K} \frac{|f_k(\bar{a})|}{|\mu_k|}\omega_k.$$

5 bounds Z

We compute the derivative

$$Df_k(\bar{a}+rw)v = \mu_k v_k - 3\lambda(\bar{a}*\bar{a}*v)_k - 6\lambda(\bar{a}*w*v)_k r - 3\lambda(w*w*v)_k r^2.$$

Next we decompose

$$Df(\bar{a}+rw) = A^\dagger + [Df(\bar{a}) - A^\dagger] + [Df(\bar{a}+rw) - Df(\bar{a})]\,,$$

where $A^\dagger$ is an approximate Jacobian defined by

$$\begin{aligned}(A^\dagger a)_F &= J_F a_F \\ (A^\dagger a)_k &= \mu_k a_k \qquad \text{for } |k| > K.\end{aligned}$$

We recall that J_F is the *exact* Jacobian of the finite dimensional map $a_F \to f_F(a_F^0)$ evaluated at $\bar{a}$, i.e. $J_F v_F = Df_F(\bar{a})v_F^0$. In this notation we have

$$[I - ADf(\bar{a}+rw)]v = \big[I - AA^\dagger\big]v - A\big[Df(\bar{a}) - A^\dagger\big]v - A\big[Df(\bar{a}+rw) - Df(\bar{a})\big]v.$$

We will estimate the terms

$$\begin{aligned}Q_0 &= \sup_{v\in B} \|[I - AA^\dagger]v\|_\nu \\ Q_1 &= \sup_{v\in B} \|A[Df(\bar{a}) - A^\dagger]v\|_\nu \\ Q_2 + rQ_3 &= \sup_{v,w\in B} \|A[Df(\bar{a}+rw) - Df(\bar{a})]v\|_\nu\end{aligned}$$

separately. Since

$$\begin{aligned}([I - AA^\dagger]v)_F &= (I - A_F J_F)v_F \\ ([I - AA^\dagger]v)_\infty &= 0\,,\end{aligned}$$

the first term can be estimated using the operator norm of the finite matrix $I - A_F J_F$:

$$\widetilde{Q}_0 = \|I - A_F J_F\|_\omega.$$

For the second term, we start by writing

$$[[Df(\bar{a}) - A^\dagger]v]_k = \begin{cases} -3\lambda m_k(\bar{a}*\bar{a}*v_\infty^0)_k & \text{for } |k| \le K \\ -3\lambda m_k(\bar{a}*\bar{a}*v)_k & \text{for } |k| > K,\end{cases}$$

where v_∞^0 is obtained from v by putting the coefficients with $|k| \le K$ equal to zero.

For $|k| \le K$ we want to bound

$$m_k|(\bar{a}*\bar{a}*v_\infty^0)_k| \le \widetilde{P}_k$$

using the *dual* estimate. This is slightly subtle, as one needs to take care of the symmetries and multiplicities:

$$\widetilde{P}_k \stackrel{\text{def}}{=} \max_{K<|n|\le 3K} \frac{m_k m_n}{4\omega_n}\big|(\bar{a}*\bar{a})_{k-(n_1,n_2)} + (\bar{a}*\bar{a})_{k-(-n_1,n_2)} + (\bar{a}*\bar{a})_{k-(n_1,-n_2)} + (\bar{a}*\bar{a})_{k-(-n_1,-n_2)}\big|.$$

This is then combined with a norm estimate for the premultiplication by A_F. This is not optimal, but a lot less work than the full dual estimate. Some of the other bounds below could also be sharpened somewhat.

The tail part $\mu_k^{-1} P_k$ for $|k| > K$ is estimated using the Banach algebra property the uniform (for $|k| > K$) bound

$$C_K = \frac{1}{(K+1)^2 - \lambda}$$

on $\frac{m_k}{\mu_k}$. This leads to (with elementwise absolute value in $|A_F|$)

$$\widetilde{Q}_1 = 3\lambda \Big[\big\| |A_F| \cdot \widetilde{P}_F \big\|_\nu + C_K \|\bar{a} * \bar{a}\|_\nu \Big].$$

The final terms are estimated by operator estimates

$$\begin{aligned} \widetilde{Q}_2 &= 6\lambda \|AM\|_\omega \|\bar{a}\|_\nu \\ \widetilde{Q}_3 &= 3\lambda \|AM\|_\omega, \end{aligned}$$

where we haver introduced the operator M acting through $(Ma)k = m_k a_k$, and the operator norm is given by

$$\|AM\|_\omega = \max\{\|A_F M_F\|_\omega, C_K\}.$$

By collecting terms we find that

$$Z(r) = r\big(\widetilde{Q}_0 + \widetilde{Q}_1\big) + r^2 \widetilde{Q}_2 + r^3 \widetilde{Q}_3.$$