A Posteriori Verification of Invariant Objects of Evolution Equations: Periodic Orbits in the Kuramoto-Sivashinsky PDE

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Abstract

In this paper, a method to compute periodic orbits of the Kuramoto-Sivashinsky PDE via rigorous numerics is presented. This is an application and an implementation of the theoretical method introduced in [1]. Using a Newton-Kantorovich type argument (the radii polynomial approach), existence of solutions is obtained in a weighted ℓ^∞ Banach space of Fourier coefficients. Once a proof of a periodic orbit is done, an associated eigenvalue problem is solved and Floquet exponents are rigorously computed, yielding proofs that some periodic orbits are unstable. Finally, a predictor-corrector continuation method is introduced to rigorously compute global smooth branches of periodic orbits. An alternative approach and independent implementation of [1] appears in [2].

Keywords

Evolution equation · periodic orbits · contraction mapping
Kuramoto-Sivashinsky · rigorous computations · interval analysis

Mathematics Subject Classification (2010)

35R20 · 47J15 · 65G40 · 65H20 · 35B32

1 Introduction

In this paper we present an implementation of the theoretical framework introduced in [1] for computer-assisted proofs of existence of invariant objects (fixed points, travelling waves, periodic orbits, attached invariant manifolds) in semilinear parabolic equations of the form

$$u_t = Lu + N(u),$$

where L is an elliptic operator and N is a semilinear operator (of lower order than L). The invariant objects that we consider are periodic orbits of the Kuramoto-Sivashinsky partial differential equation

$$\begin{cases} u_t = -\nu u_{yyyy} - u_{yy} + 2uu_y \\ u(t,y) = u(t,y+2\pi), \quad u(t,-y) = -u(t,y) \end{cases} \tag{1}$$

where $t \geq 0$ is time, $y \in [0, 2\pi]$ is the space variable and $\nu > 0$ is a fourth-order *viscosity* damping parameter. The PDE model (1) is popular to analyze weak turbulence or *spatiotemporal chaos* [3–6].

The present work accompanies papers [1] and [2]: it is an application of [1] while providing an independent implementation from the one introduced in [2]. More precisely, the choices of spaces and algorithms chosen here are different from the ones in [2], and the present paper shows how to perform continuation with respect to parameters while [2] establishes lower bounds of analyticity of solutions.

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As already mentioned in [1], the core of our study is writing down the problem of finding a periodic orbit of (1) as a zero of a functional equation of the form $\mathcal{F}(x,\nu) = 0$ (see Section 2), where $\nu > 0$ is the viscosity parameter. The operator $\mathcal{F}$ is obtained by plugging the space-time Fourier expansion of the orbit in the PDE model. The unknown $x \in X^{\boldsymbol{s}}$ is an infinite dimensional vector of space-time Fourier coefficients of the periodic orbit living in $(X^{\boldsymbol{s}}, \|\cdot\|_{\boldsymbol{s}})$ a weighted ℓ^∞ Banach space of Fourier coefficients with decay rates $\boldsymbol{s} = (s_1, s_2)$ (see Section 3). The operator $\mathcal{F}$ is not bounded (continuous) because of the presence of the differential operator $\partial_t + \nu\partial_y^4 + \partial_y^2$. To address this problem, we introduce a pre-conditioning linear operator A_ν to smooth $\mathcal{F}$, that is such that $A_\nu\mathcal{F}$ is bounded. The operator A_ν is chosen to be an approximate inverse of $D\mathcal{F}(\bar{x},\nu)$, where $\bar{x}$ is a numerical approximation of a periodic orbit of (1) at the parameter value ν. By approximate inverse, we mean that $\|I - A_\nu D\mathcal{F}(\bar{x},\nu)\|_{X^s} < 1$. The contraction mapping theorem is then applied to prove existence of a fixed point $\tilde{x}$ of

$$T_\nu(x) \stackrel{\text{def}}{=} x - A_\nu\mathcal{F}(x,\nu),$$

where the fixed point $\tilde{x}$ is the desired periodic orbit. As a by product of the method, explicit and rigorous error bounds for $\|\tilde{x} - \bar{x}\|_{\boldsymbol{s}}$ are obtained and we have results about local uniqueness.

This strategy requires an a priori setup that allows analysis and numerics to go hand in hand: the choice of function spaces, the choice of the basis functions and Galerkin projections, the analytic estimates, and the computational parameters must all work together to bound the errors due to approximation, rounding and truncation sufficiently tightly for the verification proof to go through. The goal is to provide a mathematically rigorous statement about the validity of a *concrete* numerical simulation as interpreted as an approximate solution of the original problem.

In order to prove existence of zeros of $\mathcal{F}$, we use the radii polynomial approach which was first introduced in [7], and later on used to study dynamics of PDEs [8–11]. Since we aim at solving a parameter dependent problem $\mathcal{F}(x,\nu) = 0$ our approach natural lends itself to parameter continuation methods (e.g. see [12,13]). Inspired by [14] we introduce an algorithm to compute global smooth branches of periodic orbits (see Algorithm 6.9). Once a periodic orbit is rigorously computed, an eigenvalue-eigenvector problem is solved and Floquet exponents associated the orbit are obtained.

Before proceeding any further, it is worth mentioning that we are not the first to obtain rigorous proofs of existence of solutions of (1). In the early pioneer work [15] the method of self-consistent bounds is developed and applied to obtain computer-assisted proofs of equilibria. More recently, a global bifurcation diagram of equilibria has been rigorously computed [16], where different types of bifurcations are proven. Computer-assisted proofs for periodic orbits of (1) have also been obtained. Methods based on a rigorous integration of the flow are introduced in [17–19], where the orbits are obtained by proving existence of fixed points of a Poincaré map constructed using a rigorous integrator. In [17,18] the proofs are based on the Brouwer Theorem in case of attracting orbits and on the Miranda Theorem in case of unstable ones. The symmetry of the periodic orbits is exploited when possible in order to simplify the set-up. Since the proofs are purely topological, no results about stability of the orbits are obtained. The method presented in [19] uses analyticity of the solutions, it derives estimates not only for the time-t map but also for its derivative which allows obtaining results about stability of the periodic orbits.

Our approach for computer-assisted proofs of periodic orbits of (1) has a different flavour from the above mentioned more geometric *state-space approaches*. We present a functional analytic approach which builds on the theory developed in [7,9,14]. The set-up does not require integration of the flow, does not use information about the symmetry and the stability of the orbits, and does not explicitly require *aligning windows* or finding *good coordinates*. In fact, the choice of the approximate inverse A_ν mentioned above automatically takes care of that. This *automatic* feature of our proposed approach comes however with a price as it sometimes requires computing (even though non rigorously) the inverse of a large matrix. The fact that we solve rigorously the eigenvalue problem to compute the Floquet exponents implies that we can only prove that some solutions are unstable. Extending our approach to prove that some solutions are stable is the subject of current research. The predictor-corrector continuation method that we introduce is based on the uniform contraction principle, and allows us proving existence of segments of periodic orbits of length of the order $|\Delta_\nu| \approx 10^{-5}$ (e.g. see Theorem 8.1 and Table 1). In comparison, in [18], the proofs were obtained with $|\Delta_\nu| \approx 10^{-7}$.

The paper is organized as follows. In Section 2, we define the functional equation $\mathcal{F}(x,\nu)=0$ whose solutions are the periodic orbits of (1) at the parameter value ν. In Section 3, we introduced the Banach space $X^{\boldsymbol{s}}$ in which we look for solutions of $\mathcal{F}$. In Section 4, we introduce the pre-conditioner A and design a parameter dependent fixed point operator equation $T_\nu(x)=x$ whose fixed points $x=x(\nu)$ correspond to solutions of $\mathcal{F}(x,\nu)=0$. We construct the operator with the hope that it is a uniform contraction on a segment of parameter values $[\nu_0-|\Delta_\nu|,\nu_0+|\Delta_\nu|]$, and therefore get existence of branches of periodic orbits. In Section 5, we review the basics of the radii polynomial approach and present the theory to compute global smooth branches of periodic orbits of (1). In Section 6, we present an explicit construction of the radii polynomials as well as an algorithm to compute global smooth branches of periodic orbits. In Section 7, we introduce the radii polynomial approach to prove existence of Floquet exponents. We conclude in Section 8 by presenting the results and the details about the computer-assisted proofs.

Before proceeding with the presentation of the method, let us introduce some notation.

1.1 Notation

Given $z\in\mathbb{C}$, denote by $\mathrm{conj}(z)$ its complex conjugate. We use boldface type to denote multi-indices as in $\boldsymbol{k}=(k_1,k_2)\in\mathbb{Z}^2$. Given $\boldsymbol{k},\boldsymbol{n}\in\mathbb{Z}^2$ we also use component-wise inequalities. $\boldsymbol{k}\prec\boldsymbol{n}$ means that $k_j<n_j$ for $j=1,2$. Similarly for $\boldsymbol{k}\preceq\boldsymbol{n}$, $\boldsymbol{k}\succ\boldsymbol{n}$, and $\boldsymbol{k}\succeq\boldsymbol{n}$. Throughout this paper $\boldsymbol{m}=(m_1,m_2)$ and $\boldsymbol{M}=(M_1,M_2)$ denote computational parameters such that $\boldsymbol{M}\succeq\boldsymbol{m}$. Also $\boldsymbol{s}=(s_1,s_2)$ denote the "decay rate", where each s_j is the *decay rate* on the jth-coordinate, and is such that $s_j>1$ for $j=1,2$.

2 The operator $\mathcal{F}$

Suppose that we are looking for time periodic solutions u of period τ of (1). Letting $L\stackrel{\text{def}}{=}\frac{2\pi}{\tau}$, we get that the τ-periodic solutions of (1) can be expanded using the Fourier expansion

$$u(t,y)=\sum_{\boldsymbol{k}\in\mathbb{Z}^2}c_{\boldsymbol{k}}e^{iLk_1t}e^{ik_2y}. \tag{2}$$

Since $u\in\mathbb{R}$, then for any $\boldsymbol{k}=(k_1,k_2)\in\mathbb{Z}^2$, we have that $c_{-\boldsymbol{k}}=\mathrm{conj}(c_{\boldsymbol{k}})$, where $c_{-\boldsymbol{k}}=c_{-k_1,-k_2}$. Since $u(t,-y)=-u(t,y)$, we get that $c_{k_1,-k_2}=-c_{k_1,k_2}$, for all $(k_1,k_2)\in\mathbb{Z}^2$. Hence, for every $\boldsymbol{k}=(k_1,k_2)\succeq\boldsymbol{0}$, we have the following relations

$$c_{-k_1,-k_2}=\mathrm{conj}(c_{\boldsymbol{k}})\qquad c_{k_1,-k_2}=-c_{\boldsymbol{k}}\qquad c_{-k_1,k_2}=-\mathrm{conj}(c_{\boldsymbol{k}}). \tag{3}$$

The relations in (3) imply that to describe entirely the expansion (2), one only needs to consider the $c_{\boldsymbol{k}}$ with non negative indices. From (3), we get that

$$\mathrm{Re}(c_{k_1,0})=0,k_1\geq 0,\qquad \mathrm{Im}(c_{k_1,0})=0,k_1\geq 0,\qquad \mathrm{Re}(c_{0,k_2})=0,k_2\geq 0. \tag{4}$$

Let

$$a_{\boldsymbol{k}}\stackrel{\text{def}}{=}\mathrm{Re}(c_{\boldsymbol{k}})\ \text{ and }\ b_{\boldsymbol{k}}\stackrel{\text{def}}{=}\mathrm{Im}(c_{\boldsymbol{k}}).$$

Also, since $c_{-\boldsymbol{k}}=\mathrm{conj}(c_{\boldsymbol{k}})$, then we get that for all $\boldsymbol{k}\succeq(0,0)$,

$$a_{-\boldsymbol{k}}=a_{\boldsymbol{k}}\ \text{ and }\ b_{-\boldsymbol{k}}=-b_{\boldsymbol{k}}. \tag{5}$$

Using (4), we get that $a_{k_1,0}=b_{k_1,0}=0$ for all $k_1\geq 0$ and $a_{0,k_2}=0$ for all $k_2\geq 0$. Hence, in practice, we need to consider $a=\{a_{\boldsymbol{k}}\}_{k\succeq(1,1)}$ and $b=\{b_{\boldsymbol{k}}\}_{k\succeq(0,1)}$ as variables. Since we keep the time frequency L variable, let us define the vector of unknowns x by

$$x_{\boldsymbol{k}}=\begin{cases}L, & \boldsymbol{k}=(0,0)\\ b_{\boldsymbol{k}}, & \boldsymbol{k}=(0,k_2)\succeq(0,1)\\ \begin{pmatrix}a_{\boldsymbol{k}}\\ b_{\boldsymbol{k}}\end{pmatrix}, & \boldsymbol{k}=(k_1,k_2)\succeq(1,1).\end{cases} \tag{6}$$

Defining

$$\mathcal{I} \stackrel{\text{def}}{=} \{(0,0)\} \cup \{\boldsymbol{k} = (k_1,k_2) \mid (k_1,k_2) \succeq (0,1)\}, \tag{7}$$

we define $x = \{x_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathcal{I}}$ as the infinite dimensional vector of variable uniquely determining a periodic solution of (1). Note that for each $\boldsymbol{k} \in \mathcal{I}$, $x_{\boldsymbol{k}} \in \mathbb{R}^{d(\boldsymbol{k})}$, where $d(\boldsymbol{k}) = 1$ if $\boldsymbol{k} = (0,k_2)$ for $k_2 \geq 0$, and $d(\boldsymbol{k}) = 2$ otherwise.

Given two bi-infinite sequences $a = \{a_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^2}$ and $b = \{b_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^2}$, we denote by $a * b = \{(a*b)_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^2}$ the discrete convolution product, where $a*b$ is given component-wise by

$$(a*b)_{\boldsymbol{k}} = \sum_{\boldsymbol{k}^1+\boldsymbol{k}^2=\boldsymbol{k}} a_{\boldsymbol{k}^1} b_{\boldsymbol{k}^2}. \tag{8}$$

Plugging (2) into (1) results in solving, for all $\boldsymbol{k} = (k_1,k_2) \in \mathbb{Z}^2$

$$h_{\boldsymbol{k}} \stackrel{\text{def}}{=} \mu_{\boldsymbol{k}} c_{\boldsymbol{k}} - 2 \sum_{\boldsymbol{k}^1+\boldsymbol{k}^2=\boldsymbol{k}} i\boldsymbol{k}_2^1 c_{\boldsymbol{k}^1} c_{\boldsymbol{k}^2} = \mu_{\boldsymbol{k}} c_{\boldsymbol{k}} - k_2 i (c*c)_{\boldsymbol{k}} = 0,$$

where

$$\mu_{\boldsymbol{k}} = \mu_{k_1,k_2} \stackrel{\text{def}}{=} ik_1L + \nu k_2^4 - k_2^2.$$

Using the relations (3) and considering $\boldsymbol{k} = (k_1,k_2) \succeq \mathbf{0}$, one can show that

$$h_{-k_1,-k_2} = \text{conj}(h_{\boldsymbol{k}}) \qquad h_{k_1,-k_2} = -h_{\boldsymbol{k}} \qquad h_{-k_1,k_2} = -\text{conj}(h_{\boldsymbol{k}}). \tag{9}$$

From (9), we get that

$$\text{Re}(h_{k_1,0}) = 0, k_1 \geq 0, \qquad \text{Im}(h_{k_1,0}) = 0, k_1 \geq 0, \qquad \text{Re}(h_{0,k_2}) = 0, k_2 \geq 0. \tag{10}$$

Let

$$\begin{aligned} f_{\boldsymbol{k}} &\stackrel{\text{def}}{=} \text{Re}(h_{\boldsymbol{k}}) = \left(\nu k_2^4 - k_2^2\right) a_{\boldsymbol{k}} - (k_1L) b_{\boldsymbol{k}} + 2k_2 (a*b)_{\boldsymbol{k}} \\ g_{\boldsymbol{k}} &\stackrel{\text{def}}{=} \text{Im}(h_{\boldsymbol{k}}) = (k_1L) a_{\boldsymbol{k}} + \left(\nu k_2^4 - k_2^2\right) b_{\boldsymbol{k}} - k_2[(a*a)_{\boldsymbol{k}} - (b*b)_{\boldsymbol{k}}]. \end{aligned}$$

Using (10), we get that $f_{k_1,0} = g_{k_1,0} = 0$ for all $k_1 \geq 0$ and $f_{0,k_2} = 0$ for all $k_2 \geq 0$. Also, $h_{-\boldsymbol{k}} = \text{conj}(h_{\boldsymbol{k}})$, we get that for all $\boldsymbol{k} \succeq (0,0)$,

$$f_{-\boldsymbol{k}} = f_{\boldsymbol{k}} \quad \text{and} \quad g_{-\boldsymbol{k}} = -g_{\boldsymbol{k}}.$$

This implies that in practice, we only need to solve $f = \{f_{\boldsymbol{k}}\}_{k\succeq(1,1)} = 0$ and $g = \{g_{\boldsymbol{k}}\}_{k\succeq(0,1)} = 0$.

In order to eliminate arbitrary time shift, we introduce the notion of *phase condition*. Assume that we numerically found an approximate periodic orbit $\bar{u}$. It could be at a different parameter value ν than the one we are using for the actual rigorous computation. We want to solve for $u(t,y)$ such that $u(0,t)$ lies in the hyperplane perpendicular to the direction vector $\bar{u}_t(0,y)$ and containing the point $\bar{u}(0,y)$, i.e.

$$[u(0,y) - \bar{u}(0,y)] \cdot \bar{u}_t(0,y) = 0, \tag{11}$$

where the dot product is taken in the space $L^2([0,\frac{2\pi}{L}] \times [0,2\pi])$. Equation (11) means that we are solving on a *Poincaré section* containing $\bar{u}(0,y)$ and perpendicular to $\bar{u}_t(0,y)$. Condition (11) may be relaxed by considering only finitely many Fourier coefficients in the expansions of $u(0,y)$, $\bar{u}(0,y)$ and $\bar{u}_t(0,y)$ in (11). For $\boldsymbol{m} = (m_1,m_2)$, define

$$\bar{u}^{(\boldsymbol{m})}(t,y) \stackrel{\text{def}}{=} \sum_{|k_2|<m_2} \sum_{|k_1|<m_1} \bar{c}_{\boldsymbol{k}} e^{i\bar{L}k_1t} e^{ik_2y} \qquad \text{and} \qquad u^{(\boldsymbol{m})}(t,y) \stackrel{\text{def}}{=} \sum_{|k_2|<m_2} \sum_{|k_1|<m_1} c_{\boldsymbol{k}} e^{iLk_1t} e^{ik_2y},$$

and we introduce the *Poincaré phase condition*

$$[u^{(\boldsymbol{m})}(0,y) - \bar{u}^{(\boldsymbol{m})}(0,y)] \cdot \bar{u}_t^{(\boldsymbol{m})}(0,y) = 0. \tag{12}$$

Now, using (3), we have that

$$u^{(\boldsymbol{m})}(0,y) = \sum_{|k_2|<m_2} i\left[\sum_{|k_1|<m_1} b_{k_1,k_2}\right] e^{ik_2y} = \sum_{k_2=1}^{m_2-1} -2\left[b_{0,k_2} + 2\sum_{k_1=1}^{m_1-1} b_{k_1,k_2}\right]\sin(k_2y)$$

$$\bar{u}^{(\boldsymbol{m})}(0,y) = \sum_{|k_2|<m_2} i\left[\sum_{|k_1|<m_1} \bar{b}_{k_1,k_2}\right] e^{ik_2y} = \sum_{k_2=1}^{m_2-1} -2\left[\bar{b}_{0,k_2} + 2\sum_{k_1=1}^{m_1-1} \bar{b}_{k_1,k_2}\right]\sin(k_2y)$$

$$\bar{u}_t^{(\boldsymbol{m})}(0,y) = \sum_{|k_2|<m_2} i\left[\bar{L}\sum_{|k_1|<m_1} k_1\bar{a}_{k_1,k_2}\right] e^{ik_2y} = \sum_{k_2=1}^{m_2-1} -2\left[2\bar{L}\sum_{k_1=1}^{m_1-1} k_1\bar{a}_{k_1,k_2}\right]\sin(k_2y)$$

Therefore, in Fourier space, the Poincaré phase condition (12) is equivalent to

$$\eta(x) \stackrel{\text{def}}{=} \sum_{k_2=1}^{m_2-1}\left[\left((b_{0,k_2}-\bar{b}_{0,k_2}) + 2\sum_{k_1=1}^{m_1-1}(b_{k_1,k_2}-\bar{b}_{k_1,k_2})\right)\left(\sum_{k_1=1}^{m_1-1} k_1\bar{a}_{k_1,k_2}\right)\right] = 0. \tag{13}$$

Now that the phase condition (13) is chosen, we define $\mathcal{F} = \{\mathcal{F}_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathcal{I}}$ component-wise by

$$\mathcal{F}_{\boldsymbol{k}} = \begin{cases} \eta, & \boldsymbol{k} = (0,0) \\ g_{\boldsymbol{k}}, & \boldsymbol{k} = (0,k_2) \succeq (0,1) \\ \begin{pmatrix} f_{\boldsymbol{k}} \\ g_{\boldsymbol{k}} \end{pmatrix}, & \boldsymbol{k} = (k_1,k_2) \succeq (1,1). \end{cases} \tag{14}$$

In Section 3, we show that $\frac{2\pi}{L}$-time periodic solutions $u(t,y)$ of (1) such that $\eta = 0$ is equivalent to solve

$$\mathcal{F}(x,\nu) = 0, \tag{15}$$

in a Banach space $X^{\boldsymbol{s}}$ of algebraically decaying time-space Fourier coefficients. More precisely, $X^{\boldsymbol{s}}$ is a weighed ℓ^∞ space (see Section 3). For sake of simplicity of the presentation, for $\boldsymbol{k} = (k_1,k_2) \succeq (0,1)$, let

$$R_{\boldsymbol{k}}(\nu,L) \stackrel{\text{def}}{=} \begin{cases} \nu k_2^4 - k_2^2, & \boldsymbol{k} = (0,k_2) \succeq (0,1) \\ \begin{pmatrix} \nu k_2^4 - k_2^2 & -k_1 L \\ k_1 L & \nu k_2^4 - k_2^2 \end{pmatrix}, & \boldsymbol{k} = (k_1,k_2) \succeq (1,1) \end{cases} \tag{16}$$

$$\mathcal{N}_{\boldsymbol{k}}(x) \stackrel{\text{def}}{=} \begin{cases} -(a*a)_{\boldsymbol{k}} + (b*b)_{\boldsymbol{k}}, & \boldsymbol{k} = (0,k_2) \succeq (0,1) \\ \begin{pmatrix} 2(a*b)_{\boldsymbol{k}} \\ -(a*a)_{\boldsymbol{k}} + (b*b)_{\boldsymbol{k}} \end{pmatrix}, & \boldsymbol{k} = (k_1,k_2) \succeq (1,1) \end{cases} \tag{17}$$

so that for every $\boldsymbol{k} = (k_1,k_2) \succeq (0,1)$,

$$\mathcal{F}_{\boldsymbol{k}}(x,\nu) = R_{\boldsymbol{k}}(\nu,L)x_{\boldsymbol{k}} + k_2\mathcal{N}_{\boldsymbol{k}}(x). \tag{18}$$

We now introduce a Banach space $X^{\boldsymbol{s}}$ in which we look for the solutions of (15).

3 The Banach space $X^{\boldsymbol{s}}$

Define the *one-dimensional weights* ω_k^s by

$$\omega_k^s \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } k = 0 \\ |k|^s, & \text{if } k \neq 0. \end{cases} \tag{19}$$

Given $\boldsymbol{k} = (k_1, k_2) \in \mathbb{Z}^2$ and $\boldsymbol{s} = (s_1, s_2)$, we use the one-dimensional weights to define the *two-dimensional weights*,

$$\omega_{\boldsymbol{k}}^{\boldsymbol{s}} \stackrel{\text{def}}{=} \omega_{k_1}^{s_1}\omega_{k_2}^{s_2}. \tag{20}$$

They are used to define the norm

$$\|x\|_{\boldsymbol{s}} = \sup_{\boldsymbol{k}\in\mathcal{I}} \omega_{\boldsymbol{k}}^{\boldsymbol{s}}|x_{\boldsymbol{k}}|_\infty,$$

where $|x_{\boldsymbol{k}}|_\infty$ is the sup norm of the vector $x_{\boldsymbol{k}} \in \mathbb{R}^{d(\boldsymbol{k})}$, which is one- or two-dimensional, depending on $\boldsymbol{k}$. Define the Banach space

$$X^{\boldsymbol{s}} = \left\{x = \{x_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathcal{I}} : x_{\boldsymbol{k}} \in \mathbb{R}^{d(\boldsymbol{k})} \text{ and } \|x\|_{\boldsymbol{s}} < \infty\right\}, \tag{21}$$

consisting of sequences with algebraically decaying tails according to the rate $\boldsymbol{s}$. In other words, $X^{\boldsymbol{s}}$ is an ℓ^∞ space with a weighed supremum norm.

As already mentioned in Section 2, finding $\frac{2\pi}{L}$-time periodic solutions $u(t,y)$ of (1) such that $\eta = 0$ is equivalent to solve $\mathcal{F}(x,\nu) = 0$ in $X^{\boldsymbol{s}}$. We now make this statement precise. First given $\boldsymbol{n} = (n_1, n_2)$, denote $\boldsymbol{F_n} = \{\boldsymbol{k} \in \mathcal{I} \mid \boldsymbol{k} \prec \boldsymbol{n}\}$.

Proposition 3.1. *Fix a parameter value $\nu > 0$. Consider a fixed decay rate $\boldsymbol{s} \succ (1,1)$ and assume the existence of $\boldsymbol{n} \succ (0,0)$ such that $R_{\boldsymbol{k}}(\nu, L)$ is invertible for all $\boldsymbol{k} \notin \boldsymbol{F_n}$. Then $x \in X^{\boldsymbol{s}}$ given by* (6) *satisfies $\mathcal{F}(x,\nu) = 0$ if and only if u given by*

$$u(t,y) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} (a_{\boldsymbol{k}} + ib_{\boldsymbol{k}})e^{iLk_1t}e^{ik_2y} \tag{22}$$

is a strong, real $\frac{2\pi}{L}$-periodic solution of (1)*, where the infinite dimensional vectors $a = \{a_{\boldsymbol{k}}\}_{k\succeq(1,1)}$, $b = \{b_{\boldsymbol{k}}\}_{k\succeq(0,1)}$ satisfy the symmetry conditions* (5) *and where $\eta = 0$.*

Proof. ($\Longrightarrow$) Assume $x = (L, a, b) \in X^{\boldsymbol{s}}$ given by (6) satisfies $\mathcal{F}(x,\nu) = 0$. We first show that $x \in X^{\boldsymbol{s}_0}$, for all $\boldsymbol{s}_0 \succ (1,1)$. By definition of the Banach space, we have that $x \in X^{\boldsymbol{s}} \subset X^{\boldsymbol{s}_0}$, for all $\boldsymbol{s}_0$ satisfying $(1,1) \prec \boldsymbol{s}_0 \preceq \boldsymbol{s}$. Let us show that $x \in X^{\boldsymbol{s}_0}$ for all $\boldsymbol{s}_0 \succ \boldsymbol{s}$. Note that the space

$$\ell_{\boldsymbol{s}}^\infty \stackrel{\text{def}}{=} \left\{a = \{a_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^2} \ : \ a_{\boldsymbol{k}} \in \mathbb{R} \ \text{ and } \ \sup_{\boldsymbol{k}\in\mathbb{Z}^2} \omega_{\boldsymbol{k}}^{\boldsymbol{s}}|a_{\boldsymbol{k}}| < \infty\right\} \tag{23}$$

with $\boldsymbol{s} \succ (1,1)$, is an algebra under the discrete convolution product (8), that is, given any $a, b \in \ell_{\boldsymbol{s}}^\infty$, $a * b = \{(a*b)_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^2} \in \ell_{\boldsymbol{s}}^\infty$. We refer to [9,20] for more details. Therefore, we have that each component of $\mathcal{N}_{\boldsymbol{k}}(x)$ is in $\ell_{\boldsymbol{s}}^\infty$. Since $\mathcal{F}_{\boldsymbol{k}}(x,\nu) = R_{\boldsymbol{k}}(\nu,L)x_{\boldsymbol{k}} + k_2\mathcal{N}_{\boldsymbol{k}}(x) = 0$ for all $\boldsymbol{k} \succeq (0,1)$, then

$$x_{\boldsymbol{k}} = -k_2R_{\boldsymbol{k}}(\nu,L)^{-1}\mathcal{N}_{\boldsymbol{k}}(x), \quad \text{for all } \boldsymbol{k} \notin \boldsymbol{F_n}.$$

Since each component of $\mathcal{N}(x)$ is in $\ell_{\boldsymbol{s}}^\infty$, there exists $C > 0$ such that $|\mathcal{N}_{\boldsymbol{k}}(x)| \preceq \begin{pmatrix}1\\1\end{pmatrix}\frac{C}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}$, for all $\boldsymbol{k} \in \mathcal{I}$. Applying Young's inequality with $p = 4$ and $q = \frac{4}{3}$, one gets the existence of $D > 0$ such that

$$\left|k_2R_{\boldsymbol{k}}(\nu,L)^{-1}\mathcal{N}_{\boldsymbol{k}}(x)\omega_{\boldsymbol{k}}^{(\frac{1}{2},1)}\right|_\infty \leq \left(\frac{k_1^{\frac{1}{2}}k_2^2|\nu k_2^4 - k_2^2|}{(\nu k_2^4 - k_2^2)^2 + (k_1L)^2} + \frac{k_1^{\frac{3}{2}}k_2^2L}{(\nu k_2^4 - k_2^2)^2 + (k_1L)^2}\right)\frac{C}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \leq \frac{D}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}. \tag{24}$$

Hence,

$$x = (\{x_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\boldsymbol{F_n}}, \{x_{\boldsymbol{k}}\}_{\boldsymbol{k}\notin\boldsymbol{F_n}}) = \left(\{x_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\boldsymbol{F_n}}, \left\{-k_2R_{\boldsymbol{k}}(\nu,L)^{-1}\mathcal{N}_{\boldsymbol{k}}(x)\right\}_{\boldsymbol{k}\notin\boldsymbol{F_n}}\right) \in X^{\boldsymbol{s}+(\frac{1}{2},1)}.$$

Repeating this argument inductively implies that $x \in X^{\boldsymbol{s}_0}$ for all $\boldsymbol{s}_0 \succ \boldsymbol{s}$. In particular, that shows that the coefficients $c_{\boldsymbol{k}} = a_{\boldsymbol{k}} + ib_{\boldsymbol{k}}$ decay to zero faster than any algebraic decay. Therefore, the series (22) is uniformly convergent, and the series of u_t, u_{yy}, u_{yyyy} and uu_y are also uniformly convergent. By construction, $u(t,y)$ given by (22) is a real, strong $\frac{2\pi}{L}$-periodic solution of (1).
($\Longleftarrow$) Assume that u given by (22) is a strong, real $\frac{2\pi}{L}$-periodic solution of (1). Then by construction, the corresponding $x \in X^{\boldsymbol{s}}$ given by (6) satisfies $\mathcal{F}(x,\nu) = 0$. $\square$

The next step is to design a fixed point operator whose fixed points correspond to solutions of (15).

4 The fixed point operator T_ν

For any fixed $\nu > 0$ and decay rate $\boldsymbol{s} \succ (1,1)$, Proposition 3.1 implies that computing periodic solutions of (1) is equivalent to compute solutions x of $\mathcal{F}(x,\nu)=0$ in the Banach space $X^{\boldsymbol{s}}$. Therefore, we propose a strategy to compute branches of periodic orbits of (1) by doing a rigorous continuation on $\mathcal{F}(x,\nu)=0$ in the spirit of the methods introduced in [9]. More precisely, the idea of the method is to show the existence of a branch of periodic orbits near numerical approximations in the Banach space $X^{\boldsymbol{s}}$ via a rigorous continuation method. This process begins by assuming that we found an approximate solution $\bar{x}$ of $\mathcal{F}=0$ at the parameter value ν_0 where $\mathcal{F}=\{\mathcal{F}_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathcal{I}}$ is given component-wise by (14). For a given $\boldsymbol{m}=(m_1,m_2)$, let $n=n(\boldsymbol{m}) \stackrel{\text{def}}{=} 2m_1m_2-2m_1-m_2+2$. Given $\boldsymbol{m}=(m_1,m_2)$, we define the *finite set of indices* of "sizes" $\boldsymbol{m}$ by

$$\boldsymbol{F_m} \stackrel{\text{def}}{=} \{\boldsymbol{k}\in\mathcal{I} \mid \boldsymbol{k}\prec\boldsymbol{m}\}.$$

Given $x=\{x_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ we denote its *finite part* of size $\boldsymbol{m}$ and its corresponding *infinite part* respectively by $x_{\boldsymbol{F_m}} \stackrel{\text{def}}{=} \{x_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\boldsymbol{F_m}} \in \mathbb{R}^{n(\boldsymbol{m})}$ and $x_{\boldsymbol{I_m}} \stackrel{\text{def}}{=} \{x_{\boldsymbol{k}}\}_{\boldsymbol{k}\notin\boldsymbol{F_m}}$. Now consider a *Galerkin projection* of (15) of dimension $n(\boldsymbol{m})$ given by $\mathcal{F}^{(\boldsymbol{m})} \stackrel{\text{def}}{=} \{\mathcal{F}^{(\boldsymbol{m})}_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\boldsymbol{F_m}}$, where $\mathcal{F}^{(\boldsymbol{m})}\colon \mathbb{R}^{n(\boldsymbol{m})}\times\mathbb{R}\to\mathbb{R}^{n(\boldsymbol{m})}$, is given component-wise by

$$\mathcal{F}^{(\boldsymbol{m})}_{\boldsymbol{k}}(x_{\boldsymbol{F_m}},\nu) \stackrel{\text{def}}{=} \mathcal{F}_{\boldsymbol{k}}((x_{\boldsymbol{F_m}},0_{\boldsymbol{I_m}}),\nu), \qquad \boldsymbol{k}\in\boldsymbol{F_m}, \tag{25}$$

where $\mathcal{F}_{\boldsymbol{k}}((x_{\boldsymbol{F_m}},0_{\boldsymbol{I_m}}),\nu)$ is evaluated using (14). Now suppose that at the parameter value ν_0, we numerically found $\bar{x}_{\boldsymbol{F_m}}$ such that $\mathcal{F}^{(\boldsymbol{m})}(\bar{x}_{\boldsymbol{F_m}},\nu_0)\approx 0$. We define $\bar{x} \stackrel{\text{def}}{=} (\bar{x}_{\boldsymbol{F_m}},0_{\boldsymbol{I_m}})\in X^{\boldsymbol{s}}$. Denote

$$\bar{x}_{\boldsymbol{k}} = \begin{cases} \bar{L}, & \boldsymbol{k}=(0,0) \\ \bar{b}_{\boldsymbol{k}}, & \boldsymbol{k}=(0,k_2)\succeq(0,1) \\ \begin{pmatrix}\bar{a}_{\boldsymbol{k}}\\ \bar{b}_{\boldsymbol{k}}\end{pmatrix}, & \boldsymbol{k}=(k_1,k_2)\succeq(1,1), \end{cases}$$

where $\bar{a}_{\boldsymbol{k}}=\bar{b}_{\boldsymbol{k}}=0$ for $\boldsymbol{k}\notin\boldsymbol{F_m}$. Suppose that at the parameter value ν_0, we numerically found $\dot{x}_{\boldsymbol{F_m}}$ such that $D\mathcal{F}^{(\boldsymbol{m})}(\bar{x}_{\boldsymbol{F_m}},\nu_0)\dot{x}_{\boldsymbol{F_m}} + \frac{\partial\mathcal{F}^{(\boldsymbol{m})}}{\partial\nu}(\bar{x}_{\boldsymbol{F_m}},\nu_0)\approx 0$. Defining $\dot{x} \stackrel{\text{def}}{=} (\dot{x}_{\boldsymbol{F_m}},0_{\boldsymbol{I_m}})$ we should then have that

$$\mathcal{F}(\bar{x},\nu_0)\approx 0 \quad\text{and}\quad D\mathcal{F}(\bar{x},\nu_0)\dot{x} + \frac{\partial\mathcal{F}}{\partial\nu}(\bar{x},\nu_0)\approx 0, \tag{26}$$

assuming that the Galerkin projection dimension $\boldsymbol{m}$ is taken large enough. Consider ν close to ν_0 and define $\Delta_\nu \stackrel{\text{def}}{=} \nu-\nu_0$. Denote

$$\dot{x}_{\boldsymbol{k}} = \begin{cases} \dot{L}, & \boldsymbol{k}=(0,0) \\ \dot{b}_{\boldsymbol{k}}, & \boldsymbol{k}=(0,k_2)\succeq(0,1) \\ \begin{pmatrix}\dot{a}_{\boldsymbol{k}}\\ \dot{b}_{\boldsymbol{k}}\end{pmatrix}, & \boldsymbol{k}=(k_1,k_2)\succeq(1,1), \end{cases}$$

where $\dot{a}_{\boldsymbol{k}}=\dot{b}_{\boldsymbol{k}}=0$ for $\boldsymbol{k}\notin\boldsymbol{F_m}$. We define the set of predictors by

$$x_\nu = \bar{x}+\Delta_\nu\dot{x}. \tag{27}$$

The next step is to construct a parameter dependent fixed point equation whose fixed points correspond to the zeros of $\mathcal{F}$. Let $N \stackrel{\text{def}}{=} N(\boldsymbol{M}) \stackrel{\text{def}}{=} 2M_1M_2-2M_1-M_2+2$ and consider $(\bar{x}_{\boldsymbol{F_m}},0)\in\mathbb{R}^{N(\boldsymbol{M})}$ a vector consisting of *padding* the vector $\bar{x}_{\boldsymbol{F_m}}\in\mathbb{R}^{n(\boldsymbol{m})}$ by zeros. Consider the $N(\boldsymbol{M})\times N(\boldsymbol{M})$ Jacobian matrix $D\mathcal{F}^{(\boldsymbol{M})}((\bar{x}_{\boldsymbol{F_m}},0),\nu_0)\in M_{N(\boldsymbol{M})}(\mathbb{R})$ and assume it is non-singular and let $A_{\boldsymbol{M}}\in M_{N(\boldsymbol{M})}(\mathbb{R})$ be a numerical approximation for its inverse. To define the *tail* of the operator, we need the following result.

Lemma 4.1. *Let $\boldsymbol{M}=(M_1,M_2)\succeq(m_1,m_2)$, let $\nu_0>0$ and $\Delta_\nu\in\mathbb{R}$. If*

$$M_2 > \frac{1}{\sqrt{\nu_0-|\Delta_\nu|}}, \tag{28}$$

then $R_{\boldsymbol{k}}(\nu,\bar{L})$ given by (16) is invertible for all $\boldsymbol{k}\notin\boldsymbol{F_M}$ and for all $\nu\in[\nu_0-|\Delta_\nu|,\nu_0+|\Delta_\nu|]$.

Proof. Fix $\nu_0 > 0$, $\Delta_\nu \in \mathbb{R}$. Let $\boldsymbol{M} = (M_1, M_2) \succeq (m_1, m_2)$ such that $M_2 \geq \frac{1}{\sqrt{\nu^-}} + 1$ and consider $\boldsymbol{k} = (k_1, k_2) \notin \boldsymbol{F_M}$. Consider $\nu \in [\nu_0 - |\Delta_\nu|, \nu_0 + |\Delta_\nu|]$ and define $d = \det\left(R_{\boldsymbol{k}}(\nu, \bar{L})\right) = k_1^2 \bar{L}^2 + (\nu k_2^4 - k_2^2)^2$. We conclude that $R_{\boldsymbol{k}}(\nu, \bar{L})$ is invertible by showing that $d > 0$. If $k_1 > 0$, then $d > 0$. If $k_1 = 0$, then $k_2 \geq M_2$ since $\boldsymbol{k} = (k_1, k_2) \notin \boldsymbol{F_M}$. In this case, we use (28) to conclude that $\nu k_2^2 \geq (\nu_0 - |\Delta_\nu|) M_2^2 > 1$, and therefore $\nu k_2^4 - k_2^2 > 0$. This implies that $d > 0$. □

Assuming that (28) holds, define the *parameter dependent* linear operator A_ν on sequence spaces as

$$\left[A_\nu(x)\right]_{\boldsymbol{k}} \stackrel{\text{def}}{=} \begin{cases} \left[A_{\boldsymbol{M}} x_{\boldsymbol{F_M}}\right]_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \in \boldsymbol{F_M} \\ R_{\boldsymbol{k}}(\nu, \bar{L})^{-1} x_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \notin \boldsymbol{F_M}, \end{cases} \tag{29}$$

where $R_{\boldsymbol{k}}(\nu, \bar{L})$ is given by (16). It is important to notice that the finite part ($\boldsymbol{k} \in \boldsymbol{F_M}$) of the operator A_ν depends on ν_0 only. The operator A_ν in (29) acts as an approximation for the inverse of $D\mathcal{F}(\bar{x}, \nu)$, for ν close to ν_0.

Define the Newton-like operator by

$$T_\nu(x) \stackrel{\text{def}}{=} x - A_\nu \mathcal{F}(x, \nu). \tag{30}$$

Lemma 4.2. *Consider* $\boldsymbol{M} = (M_1, M_2) \succeq (m_1, m_2)$*, let* $\nu_0 > 0$*,* $\Delta_\nu \in \mathbb{R}$ *and* $\boldsymbol{s} = (s_1, s_2) \succ (1, 1)$ *a decay rate. Assume that* $A_{\boldsymbol{M}}$ *is invertible and that* (28) *holds. Let* $\nu \in [\nu_0 - |\Delta_\nu|, \nu_0 + |\Delta_\nu|]$*. Then, the solutions of* (15) *are in one to one correspondence with the fixed points of* T_ν*. Also, we have that*

$$T_\nu : X^{\boldsymbol{s}} \to X^{\boldsymbol{s}}. \tag{31}$$

Proof. First of all, since $A_{\boldsymbol{M}}$ is invertible and since (28) holds, then the linear operator A_ν is invertible. It follows that given $x \in X^{\boldsymbol{s}}$, $\mathcal{F}(x, \nu) = 0$ if and only if $T_\nu(x) = x$. Assume that $x \in X^{\boldsymbol{s}}$. By invertibility of $A_{\boldsymbol{M}}$, for $\boldsymbol{k} \in \boldsymbol{F_M}$,

$$\sup_{\boldsymbol{k} \in \boldsymbol{F_M}} \omega_{\boldsymbol{k}}^{\boldsymbol{s}} |[T_\nu(x)]_{\boldsymbol{k}}|_\infty = \sup_{\boldsymbol{k} \in \boldsymbol{F_M}} \omega_{\boldsymbol{k}}^{\boldsymbol{s}} |x_{\boldsymbol{k}} - [A_{\boldsymbol{M}} \mathcal{F}^{(\boldsymbol{M})}(x, \nu)]_{\boldsymbol{k}}|_\infty < \infty.$$

To show that $\sup_{\boldsymbol{k} \notin \boldsymbol{F_M}} \omega_{\boldsymbol{k}}^{\boldsymbol{s}} |[T_\nu(x)]_{\boldsymbol{k}}|_\infty < \infty$, we use again Young's inequality as in (24). We conclude that $\sup_{\boldsymbol{k} \in \mathcal{I}} \omega_{\boldsymbol{k}}^{\boldsymbol{s}} |[T_\nu(x)]_{\boldsymbol{k}}|_\infty < \infty$, and therefore $T_\nu(x) \in X^{\boldsymbol{s}}$. □

5 The radii polynomial approach and rigorous continuation

The rigorous continuation method uses the radii polynomial approach, which provides a numerically efficient way to verify that the operator T_ν is a contraction on a small closed ball centered at the numerical approximation the predictors $x_\nu = \bar{x} + \Delta_\nu \dot{x}$ in $X^{\boldsymbol{s}}$. The closed ball of radius r in $X^{\boldsymbol{s}}$, centered at the origin, is given by

$$B_r(0) \stackrel{\text{def}}{=} \prod_{\boldsymbol{k} \in \mathcal{I}} \left[-\frac{r}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}, \frac{r}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}\right]^{d(\boldsymbol{k})}, \tag{32}$$

where $d(\boldsymbol{k}) = 1$ if $\boldsymbol{k} = (0, k_2)$ and $d(\boldsymbol{k}) = 2$ otherwise. The closed ball of radius r centered at the predictor x_ν defined in (27) is then

$$B_r(x_\nu) \stackrel{\text{def}}{=} x_\nu + B_r(0). \tag{33}$$

Consider now bounds $Y_{\boldsymbol{k}}$ and $Z_{\boldsymbol{k}}$ for all $\boldsymbol{k} \in \mathcal{I}$, such that

$$\left|\left[T_\nu(x_\nu) - x_\nu\right]_{\boldsymbol{k}}\right| \leq Y_{\boldsymbol{k}}(|\Delta_\nu|), \tag{34}$$

and

$$\sup_{x_1, x_2 \in B_r(0)} \left|\left[DT_\nu(x_\nu + x_1) x_2\right]_{\boldsymbol{k}}\right| \leq Z_{\boldsymbol{k}}(r, |\Delta_\nu|). \tag{35}$$

Note that the bounds $Y_{\boldsymbol{k}}$ and $Z_{\boldsymbol{k}}$ satisfying (34) and (35) can be constructed monotone increasing in $|\Delta_\nu| \geq 0$. We refer the reader to Section 6.1 for an explicit construction of the bounds $Y_{\boldsymbol{k}}(|\Delta_\nu|)$ and $Z_{\boldsymbol{k}}(r, |\Delta_\nu|)$. The proof of the following result can be found in [9].

Lemma 5.1. *Consider $\nu = \nu_0 + \Delta_\nu$. If there exists an $r > 0$ such that $\|Y + Z\|_{\boldsymbol{s}} < r$, with $Y \stackrel{\text{def}}{=} \{Y_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathcal{I}}$ and $Z \stackrel{\text{def}}{=} \{Z_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathcal{I}}$, satisfying (34) and (35), respectively, then T_ν is a contraction mapping on $B_r(x_\nu)$ with contraction constant at most $\|Y + Z\|_{\boldsymbol{s}}/r < 1$. Furthermore, there is a unique $\tilde{x}_\nu \in B_r(x_\nu)$ such that $\mathcal{F}(\tilde{x}_\nu, \nu) = 0$, and $\tilde{x}_\nu$ lies in the interior of $B_r(x_\nu)$.*

Since our proof is computer-assisted, we need to compute only finitely many of the bounds appearing in (34) and in (35). Hence, we compute asymptotic bounds for all $\boldsymbol{k} \notin \boldsymbol{F_M}$.

To obtain uniform asymptotic bounds for the $Y_{\boldsymbol{k}}$, we compute $\tilde{Y}_{\boldsymbol{M}}(|\Delta_\nu|) \geq 0$ such that

$$Y_{\boldsymbol{k}}(|\Delta_\nu|) = \tilde{Y}_{\boldsymbol{M}}(|\Delta_\nu|)\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}\mathbb{I}^{d(\boldsymbol{k})}, \qquad \text{for } \boldsymbol{k} \notin \boldsymbol{F_M}, \tag{36}$$

where $\mathbb{I}^{d(\boldsymbol{k})} = 1$ if $d(\boldsymbol{k}) = 1$, $\mathbb{I}^{d(\boldsymbol{k})} = (1,1)^T$ if $d(\boldsymbol{k}) = 2$.

To compute asymptotic bounds for the $Z_{\boldsymbol{k}}$, we decompose the set $\mathcal{I} \setminus \boldsymbol{F_M}$ in two subsets. The decomposition is based on the fact that the inverse of the tail of the operator A_ν in (29), namely $R_{\boldsymbol{k}}(\nu, \bar{L})^{-1}$ decreases faster when k_2 grows than when k_1 grows. Given $M \in \mathbb{N}$, denote

$$I_M \stackrel{\text{def}}{=} \{k \in \mathbb{N} \mid k \geq M\}.$$

Hence, recalling (7),

$$\mathcal{I} \setminus \boldsymbol{F_M} = \mathbb{N} \times I_{M_2} \cup \bigcup_{k_2=1,\ldots,M_2-1} I_{M_1} \times \{k_2\}. \tag{37}$$

Let $\boldsymbol{k} \notin \boldsymbol{F_M}$. Then $\boldsymbol{k} \in I_{M_1} \times \{k_2\}$ for some $k_2 \in \{1, \ldots, M_2 - 1\}$ or $\boldsymbol{k} \in \mathbb{N} \times I_{M_1}$. Assume there exist $\tilde{Z}_{M_1,1}(r, |\Delta_\nu|), \ldots, \tilde{Z}_{M_1,M_2-1}(r, |\Delta_\nu|) \geq 0$, and $\tilde{Z}_{\infty,M_2}(r, |\Delta_\nu|) \geq 0$ such that if $\boldsymbol{k} = (k_1, k_2) \in I_{M_1} \times \{k_2\}$, for some $k_2 \in \{1, \ldots, M_2 - 1\}$, then

$$Z_{\boldsymbol{k}}(r, |\Delta_\nu|) = Z_{k_1,k_2}(r, |\Delta_\nu|) \stackrel{\text{def}}{=} \tilde{Z}_{M_1,k_2}(r, |\Delta_\nu|)\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}\mathbb{I}^{d(\boldsymbol{k})}, \tag{38}$$

and if $\boldsymbol{k} \in \mathbb{N} \times I_{M_2}$, then

$$Z_{\boldsymbol{k}}(r, |\Delta_\nu|) = Z_{k_1,k_2}(r, |\Delta_\nu|) \stackrel{\text{def}}{=} \tilde{Z}_{\infty,M_2}(r, |\Delta_\nu|)\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}\mathbb{I}^{d(\boldsymbol{k})}. \tag{39}$$

Definition 5.2. We define the *finite radii polynomials* $\{p_{\boldsymbol{k}}(r)\}_{\boldsymbol{k}\in\boldsymbol{F_M}}$ by

$$p_{\boldsymbol{k}}(r, |\Delta_\nu|) \stackrel{\text{def}}{=} Z_{\boldsymbol{k}}(r, |\Delta_\nu|) - r\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}\mathbb{I}^{d(\boldsymbol{k})} + Y_{\boldsymbol{k}}(|\Delta_\nu|), \tag{40}$$

and the *tail radii polynomials* by

$$\tilde{p}_{M_1,k_2}(r, |\Delta_\nu|) \stackrel{\text{def}}{=} \tilde{Z}_{M_1,k_2}(r, |\Delta_\nu|) - r + \tilde{Y}_{\boldsymbol{M}}(|\Delta_\nu|), \qquad \text{for } k_2 = 1, \ldots, M_2 - 1, \tag{41}$$

and

$$\tilde{p}_{\infty,M_2}(r, |\Delta_\nu|) \stackrel{\text{def}}{=} \tilde{Z}_{\infty,M_2}(r, |\Delta_\nu|) - r + \tilde{Y}_{\boldsymbol{M}}(|\Delta_\nu|). \tag{42}$$

The proof of the following result can be found in [14].

Lemma 5.3. *Suppose that $T_\nu \in C^\ell(X^{\boldsymbol{s}}, X^{\boldsymbol{s}})$, $\ell \in \{1, 2, \ldots, \infty\}$, and suppose that the dependency in ν is also C^ℓ. If there exist $r > 0$ and Δ_ν such that $p_{\boldsymbol{k}}(r, |\Delta_\nu|) < 0$ for all $\boldsymbol{k} \in \boldsymbol{F_M}$, $\tilde{p}_{M_1,k_2}(r, |\Delta_\nu|) < 0$, for all $k_2 = 1, \ldots, M_2 - 1$ and $\tilde{p}_{\infty,M_2}(r, |\Delta_\nu|) < 0$, then there exists a C^ℓ function*

$$\tilde{x} : [\nu_0 - |\Delta_\nu|, \nu_0 + |\Delta_\nu|] \to X^{\boldsymbol{s}} : \nu \mapsto \tilde{x}(\nu)$$

such that $\mathcal{F}(\tilde{x}(\nu), \nu) = 0$ for all $\nu \in [\nu_0 - |\Delta_\nu|, \nu_0 + |\Delta_\nu|]$. Furthermore, these are the only solutions of $\mathcal{F}(x, \nu) = 0$ in the tube $\{(x, \nu) \mid x - x_\nu \in B_r(0), |\nu - \nu_0| \leq |\Delta_\nu|\}$.

Note that the explicit derivation of the formulas for the radii polynomials as defined in Definition 5.2 are postpone to Section 6 and are given in (79), (80) and (81). For the remainder of this section we assume that $\Delta_\nu \geq 0$. The case $\Delta_\nu \leq 0$ can be handled similarly.

Assume that at $(x, \nu) = (\bar{x}_0, \nu_0)$ with predictor $\bar{x}_0 + \Delta_\nu \dot{x}_0$ and step size $\Delta_\nu \geq 0$, we successfully applied Lemma 5.3, i.e. we constructed the radii polynomials based at $(x, \nu) = (\bar{x}_0, \nu_0)$ and verified that they are all simultaneously negative, say at a radius $r_0 > 0$. After this successful step, we find the corrector $\bar{x}_1$ at $\nu = \nu_1 = \nu_0 + \Delta_\nu$ using a Newton iteration, and we rebuild the radii polynomials, now based at $(x, \nu) = (\bar{x}_1, \nu_1)$. Suppose now that we have performed two succesfull continuation steps, i.e., in both steps we have found radii r_0 and r_1, respectively, where the radii polynomials are negative. We thus have two smooth solution graphs over intervals $[\nu_0, \nu_1]$ and $[\nu_1, \nu_2]$: Lemma 5.3 implies the existence of two functions $\tilde{x}^0(\nu)$ and $\tilde{x}^1(\nu)$ of class C^ℓ such that

$$\mathcal{C}_0 \stackrel{\text{def}}{=} \left\{ \left(\nu, \tilde{x}^0(\nu)\right) \mid \nu \in [\nu_0, \nu_1] \right\} \qquad \text{and} \qquad \mathcal{C}_1 \stackrel{\text{def}}{=} \left\{ \left(\nu, \tilde{x}^1(\nu)\right) \mid \nu \in [\nu_1, \nu_2] \right\}$$

are smooth branches of solutions of $\mathcal{F}(x, \nu) = 0$. The question is to determine whether or not $\mathcal{C}_0$ and $\mathcal{C}_1$ connect at the parameter value ν_1 to form a smooth continuum of zeros $\mathcal{C}_0 \cup \mathcal{C}_1$. In other words, can we prove that $\tilde{x}^0(\nu_1) = \tilde{x}^1(\nu_1)$ and that the connection is smooth? At the parameter value ν_1, we have two sets enclosing a unique zero namely

$$B_0 \stackrel{\text{def}}{=} \bar{x}_0 + (\nu_1 - \nu_0)\dot{x}_0 + B_{r_0}(0) \qquad \text{and} \qquad B_1 \stackrel{\text{def}}{=} \bar{x}_1 + B_{r_1}(0).$$

We want to prove that the solutions in B_0 and B_1 are the same. We return now to the radii polynomials constructed at basepoint $(x, \nu) = (\bar{x}_1, \nu_1)$, and evaluate them at $\Delta_\nu = 0$. Since $p_{\boldsymbol{k}}(r_1, 0) < 0$ for all $\boldsymbol{k} \in \boldsymbol{F_M}$, $\tilde{p}_{M_1, k_2}(r_1, 0) < 0$, for all $k_2 = 1, \ldots, M_2 - 1$ and $\tilde{p}_{\infty, M_2}(r_1, 0) < 0$, we can find a non empty interval $\mathcal{I}_0 \stackrel{\text{def}}{=} [r_1^-, r_1^+]$ strictly containing r_1 such that, for all $r \in \mathcal{I}_0$, we have that $p_{\boldsymbol{k}}(r, 0) < 0$ for all $\boldsymbol{k} \in \boldsymbol{F_M}$, $\tilde{p}_{M_1, k_2}(r, 0) < 0$, for all $k_2 = 1, \ldots, M_2 - 1$ and $\tilde{p}_{\infty, M_2}(r, 0) < 0$. We now have two additional sets enclosing a unique zero at parameter value ν_1, namely

$$B_{1^\pm} \stackrel{\text{def}}{=} \bar{x}_1 + B_{r_1^\pm}(0).$$

Proposition 5.4. *If $B_0 \subset B_{1^+}$ or $B_{1^-} \subset B_0$, then $\mathcal{C}_0 \cup \mathcal{C}_1$ consists of a continuous branch of solutions of $\mathcal{F}(x, \nu) = 0$, and $\mathcal{C}_0 \cap \mathcal{C}_1 = \{(\nu_1, \tilde{x}^0(\nu_1))\} = \{(\nu_1, \tilde{x}^1(\nu_1))\} \in \{\nu_1\} \times B_0 \cap B_1$. Moreover, if $T(x, \nu) \stackrel{\text{def}}{=} T_\nu(x)$ is of class C^ℓ, then $\mathcal{C}_0 \cup \mathcal{C}_1$ is a C^ℓ smooth curve.*

Proof. See Proposition 8 in [14]. □

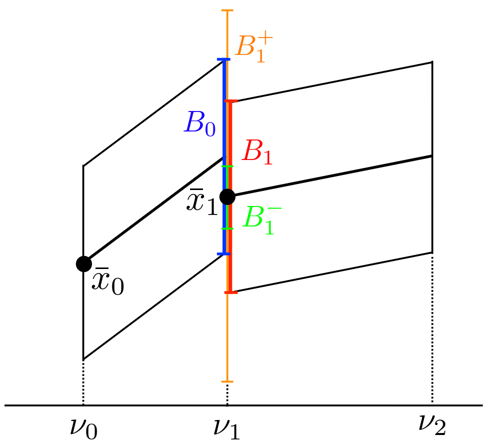

Figure 1: $B_0 \cap B_1$ contains a unique solution of (15) and $\mathcal{C}_0 \cup \mathcal{C}_1$ consists of a continuum of zeros. This picture illustrates the proof of Proposition 5.4

Remark 5.5. In practice, the hypothesis of Proposition 5.4 are verified as follows. The center points $\bar{x}_0 + (\nu_1 - \nu_0)\dot{x}_0$ of B_0 and $\bar{x}_1$ of B_1 and $B_{1\pm}$ are computed using the finite dimensional approximations $\mathcal{F}^{(\boldsymbol{m}^0)}$ and $\mathcal{F}^{(\boldsymbol{m}^1)}$ of $\mathcal{F}$, respectively, where $\mathcal{F}^{(\boldsymbol{m})}$ is given by (25). For $\boldsymbol{m}^0 = (m_1^0, m_2^0)$ and $\boldsymbol{m}^1 = (m_1^1, m_2^1)$, we define $\bar{\boldsymbol{m}} = (\bar{m}_1, \bar{m}_2)$ component-wise by $\bar{m}_j = \max\{m_j^0, m_j^1\}$, for $j = 1, 2$. Let us define the finite dimensional projections

$$B_0^{(\boldsymbol{m})} \stackrel{\text{def}}{=} (\bar{x}_0 + (\nu_1 - \nu_0)\dot{x}_0)_{\boldsymbol{F_m}} + \prod_{\boldsymbol{k} \in \boldsymbol{F_m}} \left[-\frac{r_0}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}, \frac{r_0}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \right]^{d(\boldsymbol{k})}$$

$$B_{1^-}^{(\boldsymbol{m})} \stackrel{\text{def}}{=} (\bar{x}_1)_{\boldsymbol{F_m}} + \prod_{\boldsymbol{k} \in \boldsymbol{F_m}} \left[-\frac{r_1^-}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}, \frac{r_1^-}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \right]^{d(\boldsymbol{k})} \quad \text{and} \quad B_{1^+}^{(\boldsymbol{m})} \stackrel{\text{def}}{=} (\bar{x}_1)_{\boldsymbol{F_m}} + \prod_{\boldsymbol{k} \in \boldsymbol{F_m}} \left[-\frac{r_1^+}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}, \frac{r_1^+}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \right]^{d(\boldsymbol{k})}.$$

Verifying that $B_0 \subset B_{1^+}$ (resp. $B_{1^-} \subset B_0$) is done by checking numerically that the finite dimensional box inclusion $B_0^{(\boldsymbol{m})} \subset B_{1^+}^{(\boldsymbol{m})}$ (resp. $B_{1^-}^{(\boldsymbol{m})} \subset B_0^{(\boldsymbol{m})}$) is satisfied and that $r_0 \le r_{1^+}$ (resp. $r_{1^-} \le r_0$).

6 Explicit construction of the radii polynomials

In this section, we construct the bounds required to define the radii polynomials of Definition 5.2.

6.1 The bound $\boldsymbol{Y_k}(|\boldsymbol{\Delta_\nu}|)$

The computation of the $Y_{\boldsymbol{k}}(|\Delta_\nu|)$ in (34) is done as follows. We have that $T(x_\nu) - x_\nu = -A_\nu \mathcal{F}(x_\nu, \nu)$. Let us expand $\mathcal{F}(x_\nu, \nu)$ as a polynomial in Δ_ν. Since $\mathcal{F}_{0,0}$ is linear in x and does not depend on ν,

$$\mathcal{F}_{0,0}(x_\nu, \nu) = \mathcal{F}_{0,0}(\bar{x}, \nu_0) + \Delta_\nu \left[D\mathcal{F}_{0,0}(\bar{x}, \nu_0)\dot{x} + \frac{\partial \mathcal{F}_{0,0}}{\partial \nu}(\bar{x}, \nu_0) \right].$$

For $\boldsymbol{k} \neq (0,0)$, $\mathcal{F}_{\boldsymbol{k}}(\bar{x} + \Delta_\nu \dot{x}, \nu_0 + \Delta_\nu) = y_{\boldsymbol{k}}^{(0)} + \Delta_\nu y_{\boldsymbol{k}}^{(1)} + \Delta_\nu^2 y_{\boldsymbol{k}}^{(2)}$, where

$$y_{\boldsymbol{k}}^{(0)} \stackrel{\text{def}}{=} \mathcal{F}_{\boldsymbol{k}}(\bar{x}, \nu_0) \tag{43}$$

$$y_{\boldsymbol{k}}^{(1)} \stackrel{\text{def}}{=} \left[D\mathcal{F}(\bar{x}, \nu_0)\dot{x} + \frac{\partial \mathcal{F}}{\partial \nu}(\bar{x}, \nu_0) \right]_{\boldsymbol{k}} \tag{44}$$

$$y_{\boldsymbol{k}}^{(2)} \stackrel{\text{def}}{=} k_2 \mathcal{N}_{\boldsymbol{k}}(\dot{x}), \tag{45}$$

and

$$\mathcal{N}_{\boldsymbol{k}}(\dot{x}) \stackrel{\text{def}}{=} \begin{pmatrix} 2(\dot{a} * \dot{b})_{\boldsymbol{k}} \\ -(\dot{a} * \dot{a})_{\boldsymbol{k}} + (\dot{b} * \dot{b})_{\boldsymbol{k}} \end{pmatrix}.$$

For $j = 1, 2, 3$, let

$$Y_{\boldsymbol{k}}^{(j)} \stackrel{\text{def}}{=} |A_{\boldsymbol{M}} y_{\boldsymbol{F_M}}^{(j)}|, \qquad \boldsymbol{k} \in \boldsymbol{F_M}, \tag{46}$$

which can be computed using interval arithmetic and the fast Fourier transform (e.g. see Section 2.3 in [21]). By (26), the bounds $Y_{\boldsymbol{k}}^{(0)}$ and $Y_{\boldsymbol{k}}^{(1)}$ should be small. Hence, for $\boldsymbol{k} \in \boldsymbol{F_M}$, set

$$Y_{\boldsymbol{k}}(|\Delta_\nu|) \stackrel{\text{def}}{=} \sum_{j=0}^{2} Y_{\boldsymbol{k}}^{(j)} |\Delta_\nu|^j. \tag{47}$$

To compute the uniform asymptotic bounds $\tilde{Y}_{\boldsymbol{M}}$ satisfying (36), notice first that since x_ν is such that $(x_\nu)_{\boldsymbol{k}} = 0$ for $\boldsymbol{k} \notin \boldsymbol{F_m}$, then we get that $y_{\boldsymbol{k}}^{(0)} = y_{\boldsymbol{k}}^{(1)} = y_{\boldsymbol{k}}^{(2)} = 0$ for every $\boldsymbol{k} \notin \boldsymbol{F_{\bar{M}}}$, with

$$\bar{\boldsymbol{M}} = (2m_1 - 1, 2m_2 - 1). \tag{48}$$

In order to define the asymptotic bound, we use the following result.

Lemma 6.1. *Recall that*

$$R_{\boldsymbol{k}}(\nu,\bar{L}) = \begin{pmatrix} \nu k_2^4 - k_2^2 & -k_1\bar{L} \\ k_1\bar{L} & \nu k_2^4 - k_2^2 \end{pmatrix} \quad \textit{and} \quad R_{0,k_2}(\nu,\bar{L}) = \nu k_2^4 - k_2^2.$$

Let $\nu_0, |\Delta_\nu| > 0$ *and consider* $\Lambda_{\boldsymbol{k}}$ *satisfying the component-wise bound,*

$$\sup_{\nu\in[\nu_0-|\Delta_\nu|,\nu_0+|\Delta_\nu|]} \frac{1}{(\nu k_2^4 - k_2^2)^2 + (k_1\bar{L})^2} \begin{pmatrix} |\nu k_2^4 - k_2^2| & k_1\bar{L} \\ k_1\bar{L} & |\nu k_2^4 - k_2^2| \end{pmatrix} \leq \Lambda_{\boldsymbol{k}}, \tag{49}$$

where the sup *is taken component-wise. Then, for every* $\nu \in [\nu_0 - |\Delta_\nu|, \nu_0 + |\Delta_\nu|]$,

$$\left|R_{\boldsymbol{k}}(\nu,\bar{L})^{-1}\right| \leq \Lambda_{\boldsymbol{k}}.$$

We compute $\Lambda_{\boldsymbol{k}}$ in (49) with interval arithmetic, for $j = 1,2,3$ let $\tilde{Y}_{\boldsymbol{M}}^{(j)} \stackrel{\text{def}}{=} \max_{\boldsymbol{k}\in \boldsymbol{F}_{\bar{\boldsymbol{M}}}\setminus \boldsymbol{F}_{\boldsymbol{M}}} \left\|\Lambda_{\boldsymbol{k}} y_{\boldsymbol{k}}^{(j)}\right\|_\infty \omega_{\boldsymbol{k}}^{\boldsymbol{s}}$, and let

$$\tilde{Y}_{\boldsymbol{M}}(|\Delta_\nu|) \stackrel{\text{def}}{=} \sum_{j=0}^{2} \tilde{Y}_{\boldsymbol{M}}^{(j)} |\Delta_\nu|^j. \tag{50}$$

Note that if $\boldsymbol{M} = (M_1, M_2) \succeq \bar{\boldsymbol{M}} = (2m_1 - 1, 2m_2 - 1)$, then the set $\boldsymbol{F}_{\bar{\boldsymbol{M}}} \setminus \boldsymbol{F}_{\boldsymbol{M}}$ is empty. In this case, we let $\tilde{Y}_{\boldsymbol{M}}(|\Delta_\nu|) = 0$. Hence, (36) holds for $\boldsymbol{k} \notin \boldsymbol{F}_{\boldsymbol{M}}$ since

$$\|[T(x_\nu) - x_\nu]_{\boldsymbol{k}}\|_\infty = \left\|R_{\boldsymbol{k}}(\nu,\bar{L})^{-1}\mathcal{F}_{\boldsymbol{k}}(x_\nu,\nu)\right\|_\infty \leq \left(\sum_{j=0}^{2} \left\|\Lambda_{\boldsymbol{k}} y_{\boldsymbol{k}}^{(j)}\right\|_\infty \omega_{\boldsymbol{k}}^{\boldsymbol{s}} |\Delta_\nu|^j\right) \omega_{\boldsymbol{k}}^{-\boldsymbol{s}} \leq \tilde{Y}_{\boldsymbol{M}}(|\Delta_\nu|)\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}.$$

6.2 The bound $Z_{\boldsymbol{k}}(r, |\Delta_\nu|)$

To simplify the computation of $Z_{\boldsymbol{k}}$ we introduce the operator $A_\nu^\dagger$ whose action on a vector x is

$$\left[A_\nu^\dagger(x)\right]_{\boldsymbol{k}} \stackrel{\text{def}}{=} \begin{cases} \left[D\mathcal{F}^{(\boldsymbol{M})}((\bar{x}_{\boldsymbol{F}_{\boldsymbol{m}}}, 0), \nu_0) x_{\boldsymbol{F}_{\boldsymbol{M}}}\right]_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \in \boldsymbol{F}_{\boldsymbol{M}} \\ R_{\boldsymbol{k}}(\nu,\bar{L}) x_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \notin \boldsymbol{F}_{\boldsymbol{M}}, \end{cases} \tag{51}$$

which acts as an approximate inverse for the operator A_ν. We consider the splitting

$$DT_\nu(x_\nu + x_1)x_2 = \left(I - A_\nu A_\nu^\dagger\right)x_2 - A_\nu\left(D\mathcal{F}(x_\nu + x_1, \nu) - A_\nu^\dagger\right)x_2, \tag{52}$$

where the first term is small for $\boldsymbol{k} \in \boldsymbol{F}_{\boldsymbol{M}}$, and is zero for $\boldsymbol{k} \notin \boldsymbol{F}_{\boldsymbol{M}}$. For $\boldsymbol{k} \in \boldsymbol{F}_{\boldsymbol{M}}$, we have that

$$\left|\left[\left(I - A_\nu A_\nu^\dagger\right)x_2\right]_{\boldsymbol{k}}\right| \leq \left[\left|I - A_{\boldsymbol{M}} D\mathcal{F}^{(\boldsymbol{M})}(\bar{x}_{\boldsymbol{F}_{\boldsymbol{M}}}, 0), \nu_0)\right| \omega_{\boldsymbol{F}_{\boldsymbol{M}}}^{-\boldsymbol{s}}\right]_{\boldsymbol{k}} r \tag{53}$$

where $\omega_{\boldsymbol{F}_{\boldsymbol{M}}}^{-\boldsymbol{s}} \stackrel{\text{def}}{=} \{\omega_{\boldsymbol{k}}^{-\boldsymbol{s}} \mathbb{I}^{d(\boldsymbol{k})}\}_{\boldsymbol{k}\in \boldsymbol{F}_{\boldsymbol{M}}}$, and $|\cdot|$ is the component-wise absolute values. Let $u, v \in B_1(0)$ so that $x_1 = ru$ and $x_2 = rv$. We then expand $\left[\left(D\mathcal{F}(x_\nu + x_1, \nu) - A_\nu^\dagger\right)x_2\right]_{\boldsymbol{k}}$ in terms of r and Δ_ν. Denote

$$u_{\boldsymbol{k}} = \begin{cases} L^{(u)}, & \boldsymbol{k} = (0,0) \\ b_{\boldsymbol{k}}^{(u)}, & \boldsymbol{k} = (0,k_2) \succeq (0,1) \\ \begin{pmatrix} a_{\boldsymbol{k}}^{(u)} \\ b_{\boldsymbol{k}}^{(u)} \end{pmatrix}, & \boldsymbol{k} = (k_1,k_2) \succeq (1,1) \end{cases} \quad \textit{and} \quad v_{\boldsymbol{k}} = \begin{cases} L^{(v)}, & \boldsymbol{k} = (0,0) \\ b_{\boldsymbol{k}}^{(v)}, & \boldsymbol{k} = (0,k_2) \succeq (0,1) \\ \begin{pmatrix} a_{\boldsymbol{k}}^{(v)} \\ b_{\boldsymbol{k}}^{(v)} \end{pmatrix}, & \boldsymbol{k} = (k_1,k_2) \succeq (1,1). \end{cases} \tag{54}$$

Since the phase condition (13) is linear in x and only depends on the finite modes with index $\boldsymbol{k} \in \boldsymbol{F}_{\boldsymbol{m}}$, $\left[\left(D\mathcal{F}(x_\nu + ru, \nu) - A_\nu^\dagger\right)rv\right]_{0,0} = 0$. For $\boldsymbol{k} \in \mathcal{I} \setminus \{(0,0)\}$, consider the expansion

$$\left[\left(D\mathcal{F}(x_\nu + ru, \nu) - A_\nu^\dagger\right)rv\right]_{\boldsymbol{k}} = \sum_{j=1}^{2}\sum_{l=0}^{2-j} C_{\boldsymbol{k}}^{(j,l)} r^j \Delta_\nu^l,$$

where the coefficients $C_{\boldsymbol{k}}^{(j,l)}$ are obtained through a straightforward calculation and are given by

$$C_{\boldsymbol{k}}^{(1,0)} \stackrel{\text{def}}{=} \begin{cases} 2k_2 \begin{pmatrix} (a_{\boldsymbol{I_M}}^{(v)} * \bar{b})_{\boldsymbol{k}} + (\bar{a} * b_{\boldsymbol{I_M}}^{(v)})_{\boldsymbol{k}} \\ -(a_{\boldsymbol{I_M}}^{(v)} * \bar{a})_{\boldsymbol{k}} + (\bar{b} * b_{\boldsymbol{I_M}}^{(v)})_{\boldsymbol{k}} \end{pmatrix}, & \text{if } \boldsymbol{k} \in \boldsymbol{F_M} \\ 2k_2 \begin{pmatrix} (a^{(v)} * \bar{b})_{\boldsymbol{k}} + (\bar{a} * b^{(v)})_{\boldsymbol{k}} \\ -(a^{(v)} * \bar{a})_{\boldsymbol{k}} + (\bar{b} * b^{(v)})_{\boldsymbol{k}} \end{pmatrix}, & \text{if } \boldsymbol{k} \notin \boldsymbol{F_M}, \end{cases}$$

$$C_{\boldsymbol{k}}^{(1,1)} \stackrel{\text{def}}{=} \begin{cases} k_1 L^{(v)} \begin{pmatrix} -\dot{b}_{\boldsymbol{k}} \\ \dot{a}_{\boldsymbol{k}} \end{pmatrix} + \begin{pmatrix} k_2^4 & -k_1 \dot{L} \\ k_1 \dot{L} & k_2^4 \end{pmatrix} \begin{pmatrix} a_{\boldsymbol{k}}^{(v)} \\ b_{\boldsymbol{k}}^{(v)} \end{pmatrix} + 2k_2 \begin{pmatrix} (a^{(v)} * \dot{b})_{\boldsymbol{k}} + (\dot{a} * b^{(v)})_{\boldsymbol{k}} \\ -(a^{(v)} * \dot{a})_{\boldsymbol{k}} + (\dot{b} * b^{(v)})_{\boldsymbol{k}} \end{pmatrix}, & \text{if } \boldsymbol{k} \in \boldsymbol{F_M} \\ k_1 \dot{L} \begin{pmatrix} -b_{\boldsymbol{k}}^{(v)} \\ a_{\boldsymbol{k}}^{(v)} \end{pmatrix} + 2k_2 \begin{pmatrix} (a^{(v)} * \dot{b})_{\boldsymbol{k}} + (\dot{a} * b^{(v)})_{\boldsymbol{k}} \\ -(a^{(v)} * \dot{a})_{\boldsymbol{k}} + (\dot{b} * b^{(v)})_{\boldsymbol{k}} \end{pmatrix}, & \text{if } \boldsymbol{k} \notin \boldsymbol{F_M} \end{cases}$$

and

$$C_{\boldsymbol{k}}^{(2,0)} \stackrel{\text{def}}{=} k_1 L^{(v)} \begin{pmatrix} -b_{\boldsymbol{k}}^{(u)} \\ a_{\boldsymbol{k}}^{(u)} \end{pmatrix} + k_1 L^{(u)} \begin{pmatrix} -b_{\boldsymbol{k}}^{(v)} \\ a_{\boldsymbol{k}}^{(v)} \end{pmatrix} + 2k_2 \begin{pmatrix} (a^{(v)} * b^{(u)})_{\boldsymbol{k}} + (a^{(u)} * b^{(v)})_{\boldsymbol{k}} \\ -(a^{(v)} * a^{(u)})_{\boldsymbol{k}} + (b^{(u)} * b^{(v)})_{\boldsymbol{k}} \end{pmatrix}.$$

For any k_2, the first components of $C_{0,k_2}^{(1,0)}$, $C_{0,k_2}^{(1,1)}$ and $C_{0,k_2}^{(2,0)}$ equal 0. We now compute $Z_{\boldsymbol{k}}(r, |\Delta_\nu|)$ satisfying (35) (in Section 6.2.1 for $\boldsymbol{k} \in \boldsymbol{F_M}$, and in Section 6.2.2 for $\boldsymbol{k} \notin \boldsymbol{F_M}$).

6.2.1 The bound $Z_{\boldsymbol{k}}(r, |\Delta_\nu|)$, for $\boldsymbol{k} \in \boldsymbol{F_M}$

In order to compute the bounds $Z_k(r, |\Delta_\nu|)$ for $\boldsymbol{k} \in \boldsymbol{F_M}$, we introduce intermediate upper bounds $z_{\boldsymbol{k}}^{(1,0)}, z_{\boldsymbol{k}}^{(1,1)}, z_{\boldsymbol{k}}^{(2,0)}$ such that $|C_{\boldsymbol{k}}^{(j,l)}| \leq z_{\boldsymbol{k}}^{(j,l)}$ for any $\boldsymbol{k} \in \boldsymbol{F_M}$ and for any $(j,l) \in \{(1,0), (1,1), (2,0)\}$. Defining $\omega^{(-\boldsymbol{s},a)}$ and $\omega^{(-\boldsymbol{s},b)}$ component-wise by

$$\omega_{\boldsymbol{k}}^{(-\boldsymbol{s},a)} \stackrel{\text{def}}{=} \begin{cases} 0, & \boldsymbol{k} = (0,0) \\ 0, & \boldsymbol{k} = (0,k_2) \succeq (0,1) \\ \omega_{\boldsymbol{k}}^{-\boldsymbol{s}}, & \boldsymbol{k} = (k_1,k_2) \succeq (1,1). \end{cases} \quad \textit{and} \quad \omega_{\boldsymbol{k}}^{(-\boldsymbol{s},b)} \stackrel{\text{def}}{=} \begin{cases} 0, & \boldsymbol{k} = (0,0) \\ \omega_{\boldsymbol{k}}^{-\boldsymbol{s}}, & \boldsymbol{k} = (0,k_2) \succeq (0,1) \\ \omega_{\boldsymbol{k}}^{-\boldsymbol{s}}, & \boldsymbol{k} = (k_1,k_2) \succeq (1,1), \end{cases}$$

the upper bounds $z_{\boldsymbol{k}}^{(j,l)}$ are given by

$$\begin{aligned} z_{\boldsymbol{k}}^{(1,0)} &\stackrel{\text{def}}{=} 2k_2 \begin{pmatrix} (\omega_{\boldsymbol{I_M}}^{(-\boldsymbol{s},a)} * |\bar{b}|)_{\boldsymbol{k}} + (|\bar{a}| * \omega_{\boldsymbol{I_M}}^{(-\boldsymbol{s},b)})_{\boldsymbol{k}} \\ (\omega_{\boldsymbol{I_M}}^{(-\boldsymbol{s},a)} * |\bar{a}|)_{\boldsymbol{k}} + (|\bar{b}| * \omega_{\boldsymbol{I_M}}^{(-\boldsymbol{s},b)})_{\boldsymbol{k}} \end{pmatrix} \\ z_{\boldsymbol{k}}^{(1,1)} &\stackrel{\text{def}}{=} k_1 \begin{pmatrix} |\dot{b}_{\boldsymbol{k}}| \\ |\dot{a}_{\boldsymbol{k}}| \end{pmatrix} + \begin{pmatrix} k_2^4 \omega_{\boldsymbol{k}}^{(-\boldsymbol{s},a)} + k_1 |\dot{L}| \omega_{\boldsymbol{k}}^{(-\boldsymbol{s},b)} \\ k_1 |\dot{L}| \omega_{\boldsymbol{k}}^{(-\boldsymbol{s},a)} + k_2^4 \omega_{\boldsymbol{k}}^{(-\boldsymbol{s},b)} \end{pmatrix} + 2k_2 \begin{pmatrix} (\omega^{(-\boldsymbol{s},a)} * |\dot{b}|)_{\boldsymbol{k}} + (|\dot{a}| * \omega^{(-\boldsymbol{s},b)})_{\boldsymbol{k}} \\ (\omega^{(-\boldsymbol{s},a)} * |\dot{a}|)_{\boldsymbol{k}} + (|\dot{b}| * \omega^{(-\boldsymbol{s},b)})_{\boldsymbol{k}} \end{pmatrix} \\ z_{\boldsymbol{k}}^{(2,0)} &\stackrel{\text{def}}{=} 2k_1 \begin{pmatrix} \omega_{\boldsymbol{k}}^{(-\boldsymbol{s},b)} \\ \omega_{\boldsymbol{k}}^{(-\boldsymbol{s},a)} \end{pmatrix} + 4k_2 \begin{pmatrix} 2\alpha_{k_1,s_1}^{(0,1)} \alpha_{k_2,s_2}^{(0,0)} \\ \alpha_{k_1,s_1}^{(0,0)} \alpha_{k_2,s_2}^{(0,0)} + \alpha_{k_1,s_1}^{(1,1)} \alpha_{k_2,s_2}^{(0,0)} \end{pmatrix} \omega_{\boldsymbol{k}}^{-\boldsymbol{s}}. \end{aligned}$$

Letting

$$\begin{aligned} Z_{\boldsymbol{k}}^{(1)} &\stackrel{\text{def}}{=} \left[\left| I - A_{\boldsymbol{M}} D\mathcal{F}^{(\boldsymbol{M})}((\bar{x}_{\boldsymbol{F_m}}, 0), \nu_0) \right| \omega_{\boldsymbol{F_M}}^{-\boldsymbol{s}} \right]_{\boldsymbol{k}} + \left[|A_{\boldsymbol{M}}| \left(z_{\boldsymbol{F_M}}^{(1,0)} + z_{\boldsymbol{F_M}}^{(1,1)} |\Delta_\nu| \right) \right]_{\boldsymbol{k}} \\ Z_{\boldsymbol{k}}^{(2)} &\stackrel{\text{def}}{=} \left(|A_{\boldsymbol{M}}| \, z_{\boldsymbol{F_M}}^{(2,0)} \right)_{\boldsymbol{k}} \end{aligned}$$

we set, for $\boldsymbol{k} \in \boldsymbol{F_M}$,

$$Z_k(r, |\Delta_\nu|) \stackrel{\text{def}}{=} Z_{\boldsymbol{k}}^{(2)} r^2 + Z_{\boldsymbol{k}}^{(1)} r. \tag{55}$$

Recalling (53) and using the above computations yield that for $\boldsymbol{k} \in \boldsymbol{F_M}$

$$\sup_{x_1, x_2 \in B_r(0)} \left| \left[DT_\nu(x_\nu + x_1) x_2 \right]_{\boldsymbol{k}} \right| \leq Z_{\boldsymbol{k}}(r, |\Delta_\nu|).$$

6.2.2 The bound $Z_{\boldsymbol{k}}(r,|\boldsymbol{\Delta}_\nu|)$ for $\boldsymbol{k}\notin \boldsymbol{F_M}$

We now introduce uniform bounds for $Z_{\boldsymbol{k}}(r,|\Delta_\nu|)$ for the case $\boldsymbol{k}\notin \boldsymbol{F_M}$. We begin by considering asymptotic upper bounds for $C_{\boldsymbol{k}}^{(1,0)}$, $C_{\boldsymbol{k}}^{(1,1)}$ and $C_{\boldsymbol{k}}^{(2,0)}$ for the cases $\boldsymbol{k}\notin \boldsymbol{F_M}$.

Recalling the decomposition of the set $\mathcal{I}\setminus\boldsymbol{F_M}$ in (37), we compute the uniform bounds for $Z_{\boldsymbol{k}}(r,|\Delta_\nu|)$ by distinguishing the cases $\boldsymbol{k}\in\mathbb{N}\times I_{M_2}$ or $\boldsymbol{k}\in I_{M_1}\times\{k_2\}$, for some $k_2=1,\dots,M_2-1$.

Lemma 6.2. *Let $\boldsymbol{k}\notin \boldsymbol{F_M}$ with $\boldsymbol{k}\in\mathbb{N}\times I_{M_2}$. Then*

$$|C_{\boldsymbol{k}}^{(1,0)}|\le 2k_2\begin{pmatrix}(\|\bar a\|_{\boldsymbol{s}}+\|\bar b\|_{\boldsymbol{s}})\alpha^{(0,1)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2}\\ \|\bar a\|_{\boldsymbol{s}}\alpha^{(0,0)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2}+\|\bar b\|_{\boldsymbol{s}}\alpha^{(1,1)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2}\end{pmatrix}\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}$$

$$|C_{\boldsymbol{k}}^{(1,1)}|\le k_1|\dot L|\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}\mathbb{I}^{d(\boldsymbol{k})}+2k_2\begin{pmatrix}(\|\dot a\|_{\boldsymbol{s}}+\|\dot b\|_{\boldsymbol{s}})\alpha^{(0,1)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2}\\ \|\dot a\|_{\boldsymbol{s}}\alpha^{(0,0)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2}+\|\dot b\|_{\boldsymbol{s}}\alpha^{(1,1)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2}\end{pmatrix}\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}$$

$$|C_{\boldsymbol{k}}^{(2,0)}|\le 2k_1\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}\mathbb{I}^{d(\boldsymbol{k})}+4k_2\begin{pmatrix}2\alpha^{(0,1)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2}\\ \alpha^{(0,0)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2}+\alpha^{(1,1)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2}\end{pmatrix}\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}.$$

Proof. The proof is a direct use of Lemma A.4. □

In order to present the estimates for $C_{\boldsymbol{k}}^{(i,j)}$ when $\boldsymbol{k}\in I_{M_1}\times\{k_2\}$, for some $k_2=1,\dots,M_2-1$, we need one preliminary result.

Lemma 6.3. *Let $d\in\{a^{(v)},b^{(v)}\}$, where $a^{(v)},b^{(v)}$ are components of $v=(L^{(u)},a^{(v)},b^{(v)})\in B_1(0)\subset X^{\boldsymbol{s}}$ as above. Let $\bar c\in\{\bar a,\bar b,\dot a,\dot b\}$. Let $\boldsymbol{k}\notin \boldsymbol{F_M}$ with $\boldsymbol{k}\in I_{M_1}\times\{k_2\}$ for some $k_2=1,\dots,M_2-1$. Let*

$$\Psi_{M_1,k_2}(\bar c)\stackrel{\text{def}}{=}\sum_{j_2=-m_2+1}^{m_2-1}\frac{k_2^{s_2}}{\omega_{k_2-j_2}^{s_2}}\left(|\bar c_{0,j_2}|+\sum_{j_1=1}^{m_1-1}\left[1+\frac{1}{(1-\frac{j_1}{M_1})^{s_1}}\right]|\bar c_{j_1,j_2}|.\right)\tag{56}$$

Then

$$|(d*\bar c)_{\boldsymbol{k}}|\le\Psi_{M_1,k_2}(\bar c)\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}.\tag{57}$$

Proof. Since $\bar c\in\{\bar a,\bar b,\dot a,\dot b\}$, then $\bar c_{\boldsymbol{k}}=0$ for $\boldsymbol{k}\notin\boldsymbol{F_m}$. Without loss of generality, let $d=a^{(v)}$. Hence

$$\begin{aligned}\left|(a^{(v)}*\bar c)_{\boldsymbol{k}}\right|&=\left|\sum_{(i_1+j_1,i_2+j_2)=(k_1,k_2)}a^{(v)}_{i_1,i_2}\bar c_{j_1,j_2}\right|\le\sum_{j_2=-m_2+1}^{m_2-1}\sum_{j_1=-m_1+1}^{m_1-1}\omega^{(-\boldsymbol{s},a)}_{k_1-j_1,k_2-j_2}|\bar c_{j_1,j_2}|\\&=\left[\sum_{j_2=-m_2+1}^{m_2-1}\sum_{j_1=-m_1+1}^{m_1-1}\frac{k_1^{s_1}}{(k_1-j_1)^{s_1}}\frac{k_2^{s_2}}{\omega_{k_2-j_2}^{s_2}}|\bar c_{j_1,j_2}|\right]\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}\\&=\left[\sum_{j_2=-m_2+1}^{m_2-1}\frac{k_2^{s_2}}{\omega_{k_2-j_2}^{s_2}}\sum_{j_1=-m_1+1}^{m_1-1}\frac{1}{(1-\frac{j_1}{k_1})^{s_1}}|\bar c_{j_1,j_2}|\right]\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}\le\Psi_{M_1,k_2}(\bar c)\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}.\end{aligned}$$ □

The proof of the following result follows from Lemma 6.3 and Lemma A.4.

Lemma 6.4. *For $\boldsymbol{k}\notin \boldsymbol{F_M}$ with $\boldsymbol{k}\in I_{M_1}\times\{k_2\}$, for some $k_2=1,\dots,M_2-1$.*

$$|C_{\boldsymbol{k}}^{(1,0)}|\le 2k_2\left(\Psi_{M_1,k_2}(\bar a)+\Psi_{M_1,k_2}(\bar b)\right)\mathbb{I}^2\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}$$

$$|C_{\boldsymbol{k}}^{(1,1)}|\le k_1|\dot L|\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}\mathbb{I}^2+2k_2\left(\Psi_{M_1,k_2}(\dot a)+\Psi_{M_1,k_2}(\dot b)\right)\mathbb{I}^2\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}$$

$$|C_{\boldsymbol{k}}^{(2,0)}|\le 2k_1\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}\mathbb{I}^2+4k_2\begin{pmatrix}2\alpha^{(0,1)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2}\\ \alpha^{(0,0)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2}+\alpha^{(1,1)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2}\end{pmatrix}\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}.$$

It is important to realize that the estimates for $|C_{\boldsymbol{k}}^{(i,j)}|\omega_{\boldsymbol{k}}^{\boldsymbol{s}}$ obtained in Lemma 6.2 and in Lemma 6.4 are still unbounded in (k_1,k_2) for $(i,j)=\{(1,0),(1,1),(2,0)\}$ because of the presence of the terms $|k_1|$ and $|k_2|$. However, the action of the tail $R_{\boldsymbol{k}}(\nu,\bar{L})^{-1}$ of the operator A_ν given in (29) takes care of that. More precisely, we obtain explicit constant vectors $V^{(i,j)}$ such that $|R_{\boldsymbol{k}}(\nu,\bar{L})^{-1}C_{\boldsymbol{k}}^{(i,j)}| \le V^{(i,j)}\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}$, for $(i,j)=\{(1,0),(1,1),(2,0)\}$. Letting

$$\beta_{\boldsymbol{k}}^{(1)}(\nu) \stackrel{\rm def}{=} \frac{|\nu k_2^4-k_2^2|k_2}{(\nu k_2^4-k_2^2)^2+(k_1\bar{L})^2}, \qquad \beta_{\boldsymbol{k}}^{(2)}(\nu) \stackrel{\rm def}{=} \frac{k_1k_2\bar{L}}{(\nu k_2^4-k_2^2)^2+(k_1\bar{L})^2},$$
$$\beta_{\boldsymbol{k}}^{(3)}(\nu) \stackrel{\rm def}{=} \frac{|\nu k_2^4-k_2^2|k_1}{(\nu k_2^4-k_2^2)^2+(k_1\bar{L})^2}, \qquad \beta_{\boldsymbol{k}}^{(4)}(\nu) \stackrel{\rm def}{=} \frac{k_1^2\bar{L}}{(\nu k_2^4-k_2^2)^2+(k_1\bar{L})^2},$$

we see that

$$R_{\boldsymbol{k}}(\nu,\bar{L})^{-1}k_1 = \frac{k_1}{(\nu k_2^4-k_2^2)^2+(k_1\bar{L})^2}\begin{pmatrix} |\nu k_2^4-k_2^2| & k_1\bar{L} \\ k_1\bar{L} & |\nu k_2^4-k_2^2| \end{pmatrix} = \begin{pmatrix} \beta_{\boldsymbol{k}}^{(3)}(\nu) & \beta_{\boldsymbol{k}}^{(4)}(\nu) \\ \beta_{\boldsymbol{k}}^{(4)}(\nu) & \beta_{\boldsymbol{k}}^{(3)}(\nu) \end{pmatrix}$$

and

$$R_{\boldsymbol{k}}(\nu,\bar{L})^{-1}k_2 = \frac{k_2}{(\nu k_2^4-k_2^2)^2+(k_1\bar{L})^2}\begin{pmatrix} |\nu k_2^4-k_2^2| & k_1\bar{L} \\ k_1\bar{L} & |\nu k_2^4-k_2^2| \end{pmatrix} = \begin{pmatrix} \beta_{\boldsymbol{k}}^{(1)}(\nu) & \beta_{\boldsymbol{k}}^{(2)}(\nu) \\ \beta_{\boldsymbol{k}}^{(2)}(\nu) & \beta_{\boldsymbol{k}}^{(1)}(\nu) \end{pmatrix}.$$

We now compute bounds for $\beta_{\boldsymbol{k}}^{(1)}(\nu)$, $\beta_{\boldsymbol{k}}^{(2)}(\nu)$, $\beta_{\boldsymbol{k}}^{(3)}(\nu)$ and $\beta_{\boldsymbol{k}}^{(4)}(\nu)$, for all $\boldsymbol{k} \notin \boldsymbol{F_M}$.

Lemma 6.5. *Consider $\bar{L}>0$ and $\nu>0$. Given $\boldsymbol{M}=(M_1,M_2)$, consider*

$$k_2^* \stackrel{\rm def}{=} \max_{k_2\in\{1,\dots,M_2-1\}} \left\{ k_2 \;\middle|\; |\nu k_2^4-k_2^2| \le M_1\bar{L}, \text{ for all } \nu\in[\nu_0-|\Delta_\nu|,\nu_0+|\Delta_\nu|] \right\}. \tag{58}$$

Assume that

$$\nu_0-|\Delta_\nu| > \frac{1}{(k_2^*+1)^2}. \tag{59}$$

For $k_2=1,\dots,M_2-1$, let

$$\tilde{\beta}_{M_1,k_2}^{(1)} \stackrel{\rm def}{=} \sup_{\nu\in[\nu_0-|\Delta_\nu|,\nu_0+|\Delta_\nu|]} \frac{|\nu k_2^4-k_2^2|k_2}{(\nu k_2^4-k_2^2)^2+(M_1\bar{L})^2}. \tag{60}$$

Let

$$\tilde{\beta}_{M_1,k_2}^{(2)} \stackrel{\rm def}{=} \begin{cases} \dfrac{k_2M_1\bar{L}}{((\nu_0-|\Delta_\nu|)k_2^4-k_2^2)^2+(M_1\bar{L})^2}, & k_2=1,\dots,k_2^* \\ \dfrac{1}{2k_2^3(\nu_0-|\Delta_\nu|-\frac{1}{k_2^2})}, & k_2=k_2^*+1,\dots,M_2-1 \end{cases} \tag{61}$$

and

$$\tilde{\beta}_{\infty,M_2}^{(1)} \stackrel{\rm def}{=} \frac{1}{M_2^3(\nu_0-|\Delta_\nu|-\frac{1}{M_2^2})} \quad \text{and} \quad \tilde{\beta}_{\infty,M_2}^{(2)} \stackrel{\rm def}{=} \frac{1}{2M_2^3(\nu_0-|\Delta_\nu|-\frac{1}{M_2^2})}. \tag{62}$$

Let $\boldsymbol{k}\notin\boldsymbol{F_M}$ and consider any $\nu\in[\nu_0-|\Delta_\nu|,\nu_0+|\Delta_\nu|]$. Then $\beta_{\boldsymbol{k}}^{(3)}(\nu)\le\frac{1}{2\bar{L}}$ and $\beta_{\boldsymbol{k}}^{(4)}(\nu)\le\frac{1}{\bar{L}}$. If $\boldsymbol{k}\in I_{M_1}\times\{k_2\}$ for some $k_2=1,\dots,M_2-1$, then $\beta_{\boldsymbol{k}}^{(1)}(\nu)\le\tilde{\beta}_{M_1,k_2}^{(1)}$ and $\beta_{\boldsymbol{k}}^{(2)}(\nu)\le\tilde{\beta}_{M_1,k_2}^{(1)}$. If $\boldsymbol{k}\in\mathbb{N}\times I_{M_2}$, then $\beta_{\boldsymbol{k}}^{(1)}(\nu)\le\tilde{\beta}_{\infty,M_2}^{(1)}$ and $\beta_{\boldsymbol{k}}^{(2)}(\nu)\le\tilde{\beta}_{\infty,M_2}^{(2)}$.

Proof. Consider $\boldsymbol{k}\notin\boldsymbol{F_M}$ and $\nu\in[\nu_0-|\Delta_\nu|,\nu_0+|\Delta_\nu|]$. Recall Young's inequality with $p=q=2$

$$\frac{ab}{a^2+b^2}\le\frac{1}{2}, \quad \text{for all } a,b\in\mathbb{R}. \tag{63}$$

Let $a = |\nu k_2^4 - k_2^2|$, $b = k_1 \bar{L}$ and using (63), we get

$$\beta_{\boldsymbol{k}}^{(3)}(\nu) = \frac{1}{\bar{L}} \frac{|\nu k_2^4 - k_2^2| k_1 \bar{L}}{(\nu k_2^4 - k_2^2)^2 + (k_1 \bar{L})^2} \leq \frac{1}{2\bar{L}} \quad \text{and} \quad \beta_{\boldsymbol{k}}^{(4)}(\nu) = \frac{k_1^2 \bar{L}}{(\nu k_2^4 - k_2^2)^2 + (k_1 \bar{L})^2} \leq \frac{k_1^2 \bar{L}}{(k_1 \bar{L})^2} = \frac{1}{\bar{L}}.$$

Let us compute upper bounds for $\beta_{\boldsymbol{k}}^{(1)}(\nu)$. Recall (37) and assume first that $\boldsymbol{k} \in \mathbb{N} \times I_{M_2}$. This implies that $k_2 \geq M_2$ and then

$$\beta_{\boldsymbol{k}}^{(1)}(\nu) = \frac{|\nu k_2^4 - k_2^2| k_2}{(\nu k_2^4 - k_2^2)^2 + (k_1 \bar{L})^2} \leq \frac{1}{k_2^3 |\nu - \frac{1}{k_2^2}|} \leq \beta_{\infty, M_2}^{(1)} = \frac{1}{M_2^3 (\nu_0 - |\Delta_\nu| - \frac{1}{M_2^2})}, \tag{64}$$

which is a positive upper bound. Assume now that $\boldsymbol{k} \in I_{M_1} \times \{k_2\}$ for some $k_2 = 1, \ldots, M_2 - 1$. Then $k_1 \geq M_1$. In this case,

$$\beta_{\boldsymbol{k}}^{(1)}(\nu) = \beta_{k_1,k_2}^{(1)}(\nu) \leq \beta_{M_1,k_2}^{(1)} = \sup_{\nu \in [\nu_0 - |\Delta_\nu|, \nu_0 + |\Delta_\nu|]} \frac{|\nu k_2^4 - k_2^2| k_2}{(\nu k_2^4 - k_2^2)^2 + (M_1 \bar{L})^2}. \tag{65}$$

Combining (64) and (65), we have that for any $\boldsymbol{k} \notin \boldsymbol{F_M}$, $\beta_{\boldsymbol{k}}^{(1)}(\nu) \leq \beta_{\boldsymbol{M}}^{(1)}$. We now bound $\beta_{\boldsymbol{k}}^{(2)}(\nu)$. Recall (37) and assume first that $\boldsymbol{k} \in \mathbb{N} \times I_{M_2}$, and so $k_2 \geq M_2$. Let $a = |\nu k_2^4 - k_2^2|$, $b = k_1 \bar{L}$. From (63)

$$\beta_{\boldsymbol{k}}^{(2)}(\nu) = \frac{k_1 k_2 \bar{L}}{(\nu k_2^4 - k_2^2)^2 + (k_1 \bar{L})^2} = \frac{k_2}{a} \frac{ab}{a^2 + b^2} \leq \frac{k_2}{2a} \leq \tilde{\beta}_{\infty, M_2}^{(2)} = \frac{1}{2 M_2^3 (\nu_0 - |\Delta_\nu| - \frac{1}{M_2^2})}. \tag{66}$$

Assume now that $\boldsymbol{k} \in I_{M_1} \times \{k_2\}$ for some $k_2 = 1, \ldots, M_2 - 1$. Then $k_1 \geq M_1$. One can write $\beta_{\boldsymbol{k}}^{(2)}(\nu) = k_2 f(k_1 \bar{L})$, with $f(x) = \frac{x}{\gamma + x^2}$, where $\gamma = (\nu k_2^4 - k_2^2)^2$. We have that $f(0) = \lim_{x \to \infty} f(x) = 0$ and that $f(x) > 0$ for $x > 0$. Also, $x = \sqrt{\gamma}$ is the only $x > 0$ such that $f'(x) = 0$. That implies that for every $x \geq \sqrt{\gamma}$, f is decreasing. Therefore, for any $1 \leq k_2 < M_2$ satisfying $x = k_1 \bar{L} \geq M_1 \bar{L} \geq \sqrt{\gamma} = |\nu k_2^4 - k_2^2|$,

$$\beta_{\boldsymbol{k}}^{(2)}(\nu) = k_2 f(k_1 \bar{L}) \leq k_2 f(M_1 \bar{L}) = \frac{k_2 M_1 \bar{L}}{(\nu k_2^4 - k_2^2)^2 + (M_1 \bar{L})^2}.$$

Recalling k_2^* in (58), if $k_2 \in \{1, \ldots, k_2^*\}$, then

$$\beta_{\boldsymbol{k}}^{(2)}(\nu) = \beta_{k_1,k_2}^{(2)}(\nu) \leq \tilde{\beta}_{M_1,k_2}^{(2)} = \frac{k_2 M_1 \bar{L}}{((\nu_0 - |\Delta_\nu|) k_2^4 - k_2^2)^2 + (M_1 \bar{L})^2}. \tag{67}$$

For the final cases $k_2 \in \{k_2^* + 1, \ldots, M_2 - 1\}$, we have that

$$\beta_{\boldsymbol{k}}^{(2)}(\nu) = \beta_{k_1,k_2}^{(2)}(\nu) \leq \frac{1}{2 k_2^3 (\nu_0 - |\Delta_\nu| - \frac{1}{k_2^2})} \tag{68}$$

which is a positive bound, thanks to (59). □

The proof of the following result is a direct application of Lemma 6.2 and Lemma 6.5.

Lemma 6.6. *Given* $\boldsymbol{M} = (M_1, M_2) \succeq (m_1, m_2)$, *let*

$$V_{\infty,M_2}^{(1,0)} \stackrel{\text{def}}{=} \left\| 2 \begin{pmatrix} \tilde{\beta}_{\infty,M_2}^{(1)} & \tilde{\beta}_{\infty,M_2}^{(2)} \\ \tilde{\beta}_{\infty,M_2}^{(2)} & \tilde{\beta}_{\infty,M_2}^{(1)} \end{pmatrix} \begin{pmatrix} (\|\bar{a}\|_{\boldsymbol{s}} + \|\bar{b}\|_{\boldsymbol{s}}) \alpha_{M_1,s_1}^{(0,1)} \alpha_{M_2,s_2}^{(0,0)} \\ \|\bar{a}\|_{\boldsymbol{s}} \alpha_{M_1,s_1}^{(0,0)} \alpha_{M_2,s_2}^{(0,0)} + \|\bar{b}\|_{\boldsymbol{s}} \alpha_{M_1,s_1}^{(1,1)} \alpha_{M_2,s_2}^{(0,0)} \end{pmatrix} \right\|_\infty \tag{69}$$

$$V_{\infty,M_2}^{(1,1)} \stackrel{\text{def}}{=} \left\| \frac{3|\dot{L}|}{2\bar{L}} \mathbb{I}^2 + 2 \begin{pmatrix} \tilde{\beta}_{\infty,M_2}^{(1)} & \tilde{\beta}_{\infty,M_2}^{(2)} \\ \tilde{\beta}_{\infty,M_2}^{(2)} & \tilde{\beta}_{\infty,M_2}^{(1)} \end{pmatrix} \begin{pmatrix} (\|\dot{a}\|_{\boldsymbol{s}} + \|\dot{b}\|_{\boldsymbol{s}}) \alpha_{M_1,s_1}^{(0,1)} \alpha_{M_2,s_2}^{(0,0)} \\ \|\dot{a}\|_{\boldsymbol{s}} \alpha_{M_1,s_1}^{(0,0)} \alpha_{M_2,s_2}^{(0,0)} + \|\dot{b}\|_{\boldsymbol{s}} \alpha_{M_1,s_1}^{(1,1)} \alpha_{M_2,s_2}^{(0,0)} \end{pmatrix} \right\|_\infty \tag{70}$$

$$V_{\infty,M_2}^{(2,0)} \stackrel{\text{def}}{=} \left\| \frac{3}{\bar{L}} \mathbb{I}^2 + 4 \begin{pmatrix} \tilde{\beta}_{\infty,M_2}^{(1)} & \tilde{\beta}_{\infty,M_2}^{(2)} \\ \tilde{\beta}_{\infty,M_2}^{(2)} & \tilde{\beta}_{\infty,M_2}^{(1)} \end{pmatrix} \begin{pmatrix} 2 \alpha_{M_1,s_1}^{(0,1)} \alpha_{M_2,s_2}^{(0,0)} \\ \alpha_{M_1,s_1}^{(0,0)} \alpha_{M_2,s_2}^{(0,0)} + \alpha_{M_1,s_1}^{(1,1)} \alpha_{M_2,s_2}^{(0,0)} \end{pmatrix} \right\|_\infty. \tag{71}$$

Consider $\boldsymbol{k} \notin \boldsymbol{F_M}$ *with* $\boldsymbol{k} \in \mathbb{N} \times I_{M_2}$. *Then, for* $(i,j) = \{(1,0), (1,1), (2,0)\}$,

$$\|R_{\boldsymbol{k}}(\nu, \bar{L})^{-1} C_{\boldsymbol{k}}^{(i,j)}\|_\infty \leq V_{\infty,M_2}^{(i,j)} \omega_{\boldsymbol{k}}^{-\boldsymbol{s}}. \tag{72}$$

Defining the bound

$$\tilde{Z}_{\infty,M_2}(r,|\Delta_\nu|) \stackrel{\rm def}{=} V^{(1,0)}_{\infty,M_2} r + V^{(1,1)}_{\infty,M_2}|\Delta_\nu| r + V^{(2,0)}_{\infty,M_2} r^2, \tag{73}$$

we get that for any $\boldsymbol{k} \notin \boldsymbol{F_M}$ with $\boldsymbol{k} \in \mathbb{N} \times I_{M_2}$

$$\sup_{x_1,x_2\in B_r(0)} \|[DT_\nu(x_\nu + x_1)x_2]_{\boldsymbol{k}}\|_\infty \leq \sup_{x_1,x_2\in B_r(0)} \left\| R_{\boldsymbol{k}}(\nu,\bar{L})^{-1} \sum_{j=1}^{2}\sum_{l=0}^{2-j} C^{(j,l)}_{\boldsymbol{k}} r^j \Delta^l_\nu \right\|_\infty \leq \tilde{Z}_{\infty,M_2}(r,|\Delta_\nu|)\omega^{-\boldsymbol{s}}_{\boldsymbol{k}}.$$

Lemma 6.7. *Let $\boldsymbol{k} \notin \boldsymbol{F_M}$ with $\boldsymbol{k} \in I_{M_1} \times \{k_2\}$, for some $k_2 = 1,\dots,M_2-1$. Let*

$$V^{(1,0)}_{M_1,k_2} \stackrel{\rm def}{=} 2\left(\tilde{\beta}^{(1)}_{M_1,k_2} + \tilde{\beta}^{(2)}_{M_1,k_2}\right)\left(\Psi_{M_1,k_2}(\bar{a}) + \Psi_{M_1,k_2}(\bar{b})\right) \tag{74}$$

$$V^{(1,1)}_{M_1,k_2} \stackrel{\rm def}{=} \frac{3|\dot{L}|}{2\bar{L}} + 2\left(\tilde{\beta}^{(1)}_{M_1,k_2} + \tilde{\beta}^{(2)}_{M_1,k_2}\right)\left(\Psi_{M_1,k_2}(\dot{a}) + \Psi_{M_1,k_2}(\dot{b})\right) \tag{75}$$

$$V^{(2,0)}_{M_1,k_2} \stackrel{\rm def}{=} \left\| \frac{3}{\bar{L}}\mathbb{I}^2 + 4\begin{pmatrix} \tilde{\beta}^{(1)}_{\infty,M_2} & \tilde{\beta}^{(2)}_{\infty,M_2} \\ \tilde{\beta}^{(2)}_{\infty,M_2} & \tilde{\beta}^{(1)}_{\infty,M_2}\end{pmatrix}\begin{pmatrix} 2\alpha^{(0,1)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2} \\ \alpha^{(0,0)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2} + \alpha^{(1,1)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2}\end{pmatrix} \right\|_\infty . \tag{76}$$

Then, for $(i,j) = \{(1,0),(1,1),(2,0)\}$,

$$\left\| R_{\boldsymbol{k}}(\nu,\bar{L})^{-1} C^{(i,j)}_{\boldsymbol{k}} \right\|_\infty \leq V^{(i,j)}_{M_1,k_2}\omega^{-\boldsymbol{s}}_{\boldsymbol{k}}. \tag{77}$$

Proof. The proof follows from Lemma 6.4 and Lemma 6.5. □

For each $k_2 = 1,\dots,M_2-1$, defining the bound

$$\tilde{Z}_{M_1,k_2}(r,|\Delta_\nu|) \stackrel{\rm def}{=} V^{(1,0)}_{M_1,k_2} r + V^{(1,1)}_{M_1,k_2}|\Delta_\nu| r + V^{(2,0)}_{M_1,k_2} r^2, \tag{78}$$

we get that for any $\boldsymbol{k} \notin \boldsymbol{F_M}$ with $\boldsymbol{k} \in I_{M_1} \times \{k_2\}$, for some $k_2 = 1,\dots,M_2-1$

$$\sup_{x_1,x_2\in B_r(0)} \|[DT_\nu(x_\nu + x_1)x_2]_{\boldsymbol{k}}\|_\infty \leq \sup_{x_1,x_2\in B_r(0)} \left\| R_{\boldsymbol{k}}(\nu,\bar{L})^{-1} \sum_{j=1}^{2}\sum_{l=0}^{2-j} C^{(j,l)}_{\boldsymbol{k}} r^j \Delta^l_\nu \right\|_\infty \leq \tilde{Z}_{M_1,k_2}(r,|\Delta_\nu|)\omega^{-\boldsymbol{s}}_{\boldsymbol{k}}.$$

6.3 Definition of the radii polynomials

The radii polynomials of Definition 5.2 can now be defined. Combining (47) and (55), we set

$$p_{\boldsymbol{k}}(r,|\Delta_\nu|) \stackrel{\rm def}{=} Z^{(2)}_{\boldsymbol{k}} r^2 + \left(Z^{(1)}_{\boldsymbol{k}} - \omega^{-\boldsymbol{s}}_{\boldsymbol{k}}\mathbb{I}^{d(\boldsymbol{k})}\right) r + Y_{\boldsymbol{k}}(|\Delta_\nu|), \qquad \boldsymbol{k} \in \boldsymbol{F_M}. \tag{79}$$

Using (73), let

$$\tilde{p}_{\infty,M_2}(r,|\Delta_\nu|) = \tilde{Z}_{\infty,M_2}(r,|\Delta_\nu|) - r + \tilde{Y}_{\boldsymbol{M}}(|\Delta_\nu|), \tag{80}$$

and using (78), for $k_2 = 1,\dots,M_2-1$, let

$$\tilde{p}_{M_1,k_2}(r,|\Delta_\nu|) \stackrel{\rm def}{=} \tilde{Z}_{M_1,k_2}(r,|\Delta_\nu|) - r + \tilde{Y}_{\boldsymbol{M}}(|\Delta_\nu|). \tag{81}$$

6.4 Algorithm for the rigorous computation of global smooth branches of periodic orbits for the Kuramoto-Sivashinsky equation

Before presenting the general algorithm, we introduce a short algorithm that automatically generates the value of $\boldsymbol{M} = (M_1, M_2)$ before attempting to prove a piece of branch.

Algorithm 6.8 (**A priori choice for $\boldsymbol{M} = (M_1, M_2)$**). *At the parameter value ν_0, let $\bar{x}_{\boldsymbol{F_m}} = (\bar{L}, \bar{a}, \bar{b}) \in \mathbb{R}^{n(\boldsymbol{m})}$ with $n = n(\boldsymbol{m}) \stackrel{\text{def}}{=} 2m_1m_2 - 2m_1 - m_2 + 2$ an approximate solution satisfying $\mathcal{F}^{(\boldsymbol{m})}(x_{\boldsymbol{F_m}}, \nu_0) \approx 0$. Fix a decay rate $\boldsymbol{s} = (s_1, s_2)$.*

1. *(Determining M_2) Compute the analytic estimates $\alpha^{(0,0)}_{\infty,s_1}$, $\alpha^{(0,1)}_{\infty,s_1}$, $\alpha^{(1,1)}_{\infty,s_1}$ and $\alpha^{(0,0)}_{\infty,s_2}$ using* (136)*. Recalling* (20)*, compute*

$$\|\bar{a}\|_{\boldsymbol{s}} = \sup_{\boldsymbol{k}\in\mathcal{I}} |\bar{a}_{\boldsymbol{k}}|\omega^{\boldsymbol{s}}_{\boldsymbol{k}} \quad \text{and} \quad \|\bar{b}\|_{\boldsymbol{s}} = \sup_{\boldsymbol{k}\in\mathcal{I}} |\bar{b}_{\boldsymbol{k}}|\omega^{\boldsymbol{s}}_{\boldsymbol{k}}.$$

   Recalling condition (28)*, initiate the process by fixing $M_2 = \max(m_2, \frac{1}{\sqrt{\nu_0}})$. Compute $\tilde{\beta}^{(1)}_{\infty,M_2}$ and $\tilde{\beta}^{(2)}_{\infty,M_2}$ defined by* (62) *by letting $\Delta_\nu = 0$. Compute $V^{(1,0)}_{\infty,M_2}$ defined in* (69) *by using $\alpha^{(0,0)}_{\infty,s_1}$, $\alpha^{(0,1)}_{\infty,s_1}$, $\alpha^{(1,1)}_{\infty,s_1}$ and $\alpha^{(0,0)}_{\infty,s_2}$ instead of $\alpha^{(0,0)}_{M_1,s_1}$, $\alpha^{(0,1)}_{M_1,s_1}$, $\alpha^{(1,1)}_{M_1,s_1}$ and $\alpha^{(0,0)}_{M_2,s_2}$. If*

$$V^{(1,0)}_{\infty,M_2} < 0.95, \tag{82}$$

   then the choice of M_2 is done. If (82) *does not hold, then increase the dimension of M_2 by one, recompute $\tilde{\beta}^{(1)}_{\infty,M_2}$, $\tilde{\beta}^{(2)}_{\infty,M_2}$ and $V^{(1,0)}_{\infty,M_2}$ and check if* (82) *holds. Continue this process until condition* (82) *is satisfied. This process necessarily terminates.*

2. *(Determining M_1) Consider M_2 the value obtained from Step 1. Initiate the process of choosing M_1 by setting $M_1 = m_1$. Compute k_2^* satisfying* (58)*, that is*

$$k_2^* \stackrel{\text{def}}{=} \max_{k_2\in\{1,\dots,M_2-1\}} \left\{k_2 \;\middle|\; |\nu_0 k_2^4 - k_2^2| \le M_1\bar{L}\right\}.$$

   For $k_2 = 1, \dots, M_2-1$, compute $\tilde{\beta}^{(1)}_{M_1,k_2}$ and $\tilde{\beta}^{(2)}_{M_1,k_2}$ given respectively by (60) *and* (61) *with $\Delta_\nu = 0$, that is*

$$\tilde{\beta}^{(1)}_{M_1,k_2} = \frac{|\nu_0 k_2^4 - k_2^2|k_2}{(\nu_0 k_2^4 - k_2^2)^2 + (M_1\bar{L})^2}$$

$$\tilde{\beta}^{(2)}_{M_1,k_2} = \begin{cases} \dfrac{k_2 M_1 \bar{L}}{(\nu_0 k_2^4 - k_2^2)^2 + (M_1\bar{L})^2}, & k_2 = 1, \dots, k_2^* \\ \dfrac{1}{2k_2^3(\nu_0 - \frac{1}{k_2^2})}, & k_2 = k_2^* + 1, \dots, M_2 - 1. \end{cases}$$

   For $k_2 = 1, \dots, M_2-1$, compute $\Psi_{M_1,k_2}(\bar{a})$ and $\Psi_{M_1,k_2}(\bar{b})$ using the formula (56)*. Compute $V^{(1,0)}_{M_1,k_2}$ given by* (74)*. If*

$$\max_{k_2=1,\dots,M_2-1} V^{(1,0)}_{M_1,k_2} < 0.95, \tag{83}$$

   then the choice of M_1 is done. If condition (83) *does not hold, then we increase M_1 by one until condition* (83) *is satisfied. This process necessarily terminates.*

We now present a general algorithm for the rigorous computation of global smooth branches of periodic orbits for the Kuramoto-Sivashinsky equation (1).

Algorithm 6.9 (**Computing global branches of periodic orbits**). *To compute rigorously global smooth branches of periodic orbits of the Kuramoto-Sivashinsky equation* (1) *on the parameter range $[\nu_{min}, \nu_{max}]$, we proceed as follows.*

1. *Choose a minimum step-size $\Delta_{\min} > 0$ and set the maximum step-size $\Delta_{\max} = \frac{1}{2}(\nu_{max} - \nu_{min})$. Initiate a decay rate $\boldsymbol{s} = (s_1, s_2)$ and a projection dimension $\boldsymbol{m} = (m_1, m_2)$. Choose the initial parameter value $\nu_0 = \nu_{min}$ with an initial step size $\Delta_\nu > 0$, or choose the initial parameter value $\nu_0 = \nu_{max}$ with an initial step size $\Delta_\nu < 0$. The initial step size Δ_ν is chosen such that $|\Delta_\nu| \in [\Delta_{\min}, \Delta_{\max}]$. Initiate a temporary step size $\Delta^0_\nu = 0$, an initial predictor $\hat{x}_{\boldsymbol{F_m}}$ of $\mathcal{F}^{(\boldsymbol{m})}(x_{\boldsymbol{F_m}}, 0)$ and an initial radius $r_0 = 0$.*

2. *Initiate* $B_0 = B_{r_0}(\hat{x})$, *where* $\hat{x} \stackrel{\text{def}}{=} (\hat{x}_{\boldsymbol{F_m}}, 0_{\boldsymbol{I_m}})$.

3. *With a Newton-like iterative scheme, find near* $\hat{x}_{\boldsymbol{F_m}}$ *an approximate solution* $\bar{x}_{\boldsymbol{F_m}}$ *of* $\mathcal{F}^{(\boldsymbol{m})}(x_{\boldsymbol{F_m}}, \nu_0) = 0$. *Using Algorithm 6.8, determine automatically the value* $\boldsymbol{M} = (M_1, M_2) \succeq (m_1, m_2)$ *(this choice of* $\boldsymbol{M}$ *ensures that the tail radii polynomials succeed for small* $|\Delta_\nu|$*). Calculate an approximate solution* $\dot{x}_{\boldsymbol{F_m}}$ *of* $D\mathcal{F}^{(\boldsymbol{m})}(\bar{x}_{\boldsymbol{F_m}}, \nu_0)\dot{x}_{\boldsymbol{F_m}} + \frac{\partial \mathcal{F}^{(\boldsymbol{m})}}{\partial \nu}(\bar{x}_{\boldsymbol{F_m}}, \nu_0) = 0$.

4. *Compute the coefficients of the radii polynomials in* (79), (80) *and* (81).

5. *Let*

   $$\mathcal{I}_0 \stackrel{\text{def}}{=} \left\{ r \geq 0 \,|\, p_{\boldsymbol{k}}(r, 0) < 0, \ \forall\, \boldsymbol{k} \in \boldsymbol{F_M}, \tilde{p}_{\infty, M_2}(r, 0) < 0 \text{ and } \tilde{p}_{M_1, k_2}(r, 0) < 0, \ \forall\, k_2 = 1, \dots, M_2 - 1 \right\}.$$

   If $\mathcal{I}_0 = \emptyset$ *increase* m_1, m_2 *or change the value of the decay rate* $\boldsymbol{s} = (s_1, s_2)$ *and go back to Step 3. If* $\mathcal{I}_0 \neq \emptyset$, *compute* $0 < r_1^- < r_1^+$ *such that* $\{r_1^-, r_1^+\} \subset \mathcal{I}_0$. *Consider* $B_{1^-} \stackrel{\text{def}}{=} B_{r_1^-}(\bar{x})$ *and* $B_{1^+} \stackrel{\text{def}}{=} B_{r_1^+}(\bar{x})$, *where* $\bar{x} = (\bar{x}_{\boldsymbol{F_m}}, 0_{\boldsymbol{I_m}})$. *Using Remark 5.5, verify that* $B_0 \subset B_{1^+}$ *or* $B_{1^-} \subset B_0$.

6. *Let*

   $$\begin{aligned} \mathcal{I} =& [I_-, I_+] \\ \stackrel{\text{def}}{=}& \left\{ r \geq 0 \,|\, p_{\boldsymbol{k}}(r, |\Delta_\nu|) < 0, \ \forall \boldsymbol{k} \in \boldsymbol{F_M}, \tilde{p}_{\infty, M_2}(r, |\Delta_\nu|) < 0, \tilde{p}_{M_1, k_2}(r, |\Delta_\nu|) < 0, \ \forall k_2 = 1, \dots, M_2 - 1 \right\}. \end{aligned}$$

   - *If* $\mathcal{I} = \emptyset$ *then go to Step 8.*
   - *If* $\mathcal{I} \neq \emptyset$ *then let* $r = \frac{I_- + I_+}{2}$. *If, computing with interval arithmetic, one can verify that* $p_{\boldsymbol{k}}(r, |\Delta_\nu|) < 0$ *for all* $\boldsymbol{k} \in \boldsymbol{F_M}$, *that* $\tilde{p}_{M_1, k_2}(r, |\Delta_\nu|) < 0$ *for all* $k_2 = 1, \dots, M_2 - 1$, *and* $\tilde{p}_{\infty, M_2}(r, |\Delta_\nu|) < 0$, *then go to Step 7; else go to Step 8.*

7. *Update* $\Delta_\nu^0 \leftarrow \Delta_\nu$ *and* $r_0 \leftarrow r$. *If* $\frac{10}{9}|\Delta_\nu| \leq \Delta_{\max}$ *then update* $\Delta_\nu \leftarrow \frac{10}{9}\Delta_\nu$ *and go to Step 6; else go to Step 9.*

8. *If* $\Delta_\nu^0 \neq 0$ *then go to Step 9; else if* $\frac{9}{10}|\Delta_\nu| \geq \Delta_{\min}$ *then update* $\Delta_\nu \leftarrow \frac{9}{10}\Delta_\nu$ *and go to Step 6; else go to Step 10.*

9. *The continuation step has succeeded. Store, for future reference,* $\bar{x}_{\boldsymbol{F_m}}$, $\dot{x}_{\boldsymbol{F_m}}$, r_0, ν_0 *and* Δ_ν^0. *Let* $\nu_1 = \nu_0 + \Delta_\nu^0$. *Make the updates* $\nu_0 \leftarrow \nu_1$, $\Delta_\nu \leftarrow \Delta_\nu^0$, $\hat{x}_{\boldsymbol{F_m}} \leftarrow \bar{x}_{\boldsymbol{F_m}} + \Delta_\nu^0 \dot{x}_{\boldsymbol{F_m}}$ *and* $\Delta_\nu^0 \leftarrow 0$. *Update* $B_0 \leftarrow B_{r_0}(\hat{x})$ *and go to Step 3 for the next continuation step.*

10. *The continuation step has failed. Either decrease* $\Delta_{\min}$ *and return to Step 8; or increase some of the components of* $\boldsymbol{m}$ *and return to Step 3, or terminate the algorithm* unsuccessfully *at* $\nu = \nu_0$

7 Computing Floquet exponents: an eigenvalue problem

Define the right-hand side of the Kuramoto-Sivashinsky equation (1) as

$$E(u) = -\nu u_{yyyy} - u_{yy} + 2uu_y. \tag{84}$$

Assume that at a given parameter value $\nu > 0$, we have proven the existence of $\tilde{u}$ a periodic orbit of (1) using the theory of the previous sections. Assume that $\tilde{u}(t + p, y) = \tilde{u}(t, y)$ for all $t \in \mathbb{R}$ and for all $y \in [0, 2\pi]$. At this point we do not require p to be the minimal period of the periodic orbit.

Linearize (84) about $\tilde{u}$ and consider the linear system with periodic coefficients

$$\dot{\Phi} = DE(\tilde{u})\Phi. \tag{85}$$

More explicitly, one has that $DE(\tilde{u})\phi = -\nu\phi_{yyyy} - \phi_{yy} + 2(\tilde{u}\phi_y + \tilde{u}_y\phi)$. Let us parametrize the invariant normal bundle of the periodic orbit associated to the eigenvalue λ by v. Then v satisfies the equation

$$\Phi v(0, y) = e^{\lambda t} v(t, y). \tag{86}$$

Differentiating equation (86) and using (85), we obtain the *invariance equation*

$$v_t + \lambda v = DE(\tilde{u})v = -\nu v_{yyyy} - v_{yy} + 2(v_y \tilde{u} + \tilde{u}_y v). \tag{87}$$

A solution $(\lambda, v(t,y))$ of the invariance equation (90) is called an *eigenvalue-eigenvector pair.*

Remark 7.1. As mentioned above, p is not necessarily the minimal period of the periodic orbit. This is important because sometimes, we want to compute the eigenvector v as a 2τ-periodic function in time with τ the minimal period of the orbit. This is useful when computing eigenvectors parameterizing non orientable normal (un)stable bundles.

Definition 7.2. If p is the minimal period of the periodic orbit, the number λ is called a *Floquet exponent.*

Remark 7.3. Consider an eigenvalue-eigenvector pair $(\lambda, v(t,y))$ of the invariance equation (90) such that $v(t+p, y) = v(t,y)$ for all t and $x \in [0, 2\pi]$ for some $p > 0$. If $|e^{\lambda p}| > 1$, then the periodic orbit is unstable.

Let v a p-periodic function in time and 2π-periodic in space. Hence, one can expand v as follows

$$v(t,y) = \sum_{\boldsymbol{k}=(k_1,k_2)\in\mathbb{Z}^2} v_{\boldsymbol{k}} e^{i\tilde{L}k_1 t} e^{ik_2 y}. \tag{88}$$

Denote the coefficients of the solution $\tilde{u}$ by $(\tilde{L}, \tilde{a}, \tilde{b})$ with $p = 2\pi/\tilde{L}$, and consider the space-time Fourier expansion of $\tilde{u}$ given by

$$\tilde{u}(t,y) = \sum_{\boldsymbol{k}=(k_1,k_2)\in\mathbb{Z}^2} \tilde{u}_{\boldsymbol{k}} e^{i\tilde{L}k_1 t} e^{ik_2 y}, \qquad \tilde{u}_{\boldsymbol{k}} = \tilde{a}_{\boldsymbol{k}} + i\tilde{b}_{\boldsymbol{k}}. \tag{89}$$

Plugging the Fourier expansions (88) of v and (89) of $\tilde{u}$ in the invariance equation (90), one obtains that

$$\begin{aligned} h_{\boldsymbol{k}}(\lambda, v) &\stackrel{\text{def}}{=} \left(k_1\tilde{L}i + \lambda + \nu k_2^4 - k_2^2\right) v_{\boldsymbol{k}} - 2i \sum_{\boldsymbol{k}^1+\boldsymbol{k}^2=\boldsymbol{k}} \boldsymbol{k}_2^1 (v_{\boldsymbol{k}^1}\tilde{u}_{\boldsymbol{k}^2} + \tilde{u}_{\boldsymbol{k}^1} v_{\boldsymbol{k}^2}) \\ &= \left(k_1\tilde{L}i + \lambda + \nu k_2^4 - k_2^2\right) v_{\boldsymbol{k}} - 2k_2 i \sum_{\boldsymbol{k}^1+\boldsymbol{k}^2=\boldsymbol{k}} v_{\boldsymbol{k}^1}\tilde{u}_{\boldsymbol{k}^2}. \end{aligned} \tag{90}$$

Since $v \in \mathbb{R}$, then for any $\boldsymbol{k} = (k_1, k_2) \in \mathbb{Z}^2$, one has that $v_{-\boldsymbol{k}} = \text{conj}(v_{\boldsymbol{k}})$, where $v_{-\boldsymbol{k}} = v_{-k_1,-k_2}$. Since $v(t,-y) = -v(t,y)$, we get that $v_{k_1,-k_2} = -v_{k_1,k_2}$, for all $(k_1,k_2) \in \mathbb{Z}^2$. Hence, for every $\boldsymbol{k} = (k_1,k_2) \geq \mathbf{0}$, we have the following relations

$$v_{-k_1,-k_2} = \text{conj}(v_{\boldsymbol{k}}) \qquad v_{k_1,-k_2} = -v_{\boldsymbol{k}} \qquad v_{-k_1,k_2} = -\text{conj}(v_{\boldsymbol{k}}). \tag{91}$$

The relations (91) imply that to describe entirely the expansion of the eigenvector v, one only needs to consider the $v_{\boldsymbol{k}}$ with non negative indices. From (91), we get that

$$Re(v_{k_1,0}) = 0, k_1 \geq 0, \qquad Im(v_{k_1,0}) = 0, k_1 \geq 0, \qquad Re(v_{0,k_2}) = 0, k_2 \geq 0. \tag{92}$$

More explicitly, let

$$c_{\boldsymbol{k}} \stackrel{\text{def}}{=} Re(v_{\boldsymbol{k}}) \ \text{ and } \ d_{\boldsymbol{k}} \stackrel{\text{def}}{=} Im(v_{\boldsymbol{k}}),$$

Using (4), we get that $c_{k_1,0} = d_{k_1,0} = 0$ for all $k_1 \geq 0$ and $c_{0,k_2} = 0$ for all $k_2 \geq 0$. Hence, in practice, we need to consider $c = \{c_{\boldsymbol{k}}\}_{k \succeq (1,1)}$ and $d = \{d_{\boldsymbol{k}}\}_{k \succeq (0,1)}$ as variables. Since the eigenvalue λ is a variable, let us define the vector of unknowns x by

$$x_{\boldsymbol{k}} = \begin{cases} \lambda, & \boldsymbol{k} = (0,0) \\ d_{\boldsymbol{k}}, & \boldsymbol{k} = (0,k_2) \succeq (0,1) \\ \begin{pmatrix} c_{\boldsymbol{k}} \\ d_{\boldsymbol{k}} \end{pmatrix}, & \boldsymbol{k} = (k_1,k_2) \succeq (1,1). \end{cases} \tag{93}$$

Recalling the definition of the set of indices $\mathcal{I}$ in (7), one set $x = \{x_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathcal{I}}$. Hence, finding the eigenvector $v(t,y)$ of (88) corresponds to finding infinite dimensional vectors of the form $x = \{x_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathcal{I}}$ given by (93).

Eigenpairs (λ, v) of (88) come in family as $(\lambda, \alpha v)$ is also a solution of (88) for any $\alpha \in \mathbb{R}$. We therefore impose a phase condition in order to apply a contraction mapping argument. We fix the length of the eigenvector at time $t = 0$ to be approximately equal to 1 by imposing the condition

$$\eta(x) \stackrel{\text{def}}{=} \sum_{|\boldsymbol{k}|\le 3} v_{\boldsymbol{k}}^2 - 1 = \sum_{|\boldsymbol{k}|\le 3} (c_{\boldsymbol{k}}^2 + d_{\boldsymbol{k}}^2) - 1 = 0. \tag{94}$$

We solve for solutions of (90) satisfying $\eta(x) = 0$. Let

$$f_{\boldsymbol{k}} \stackrel{\text{def}}{=} Re(h_{\boldsymbol{k}}(\lambda, v)) = \left(\lambda + \nu k_2^4 - k_2^2\right) c_{\boldsymbol{k}} - k_1 \tilde{L} d_{\boldsymbol{k}} + 2k_2 \sum_{\boldsymbol{k}^1+\boldsymbol{k}^2=\boldsymbol{k}} \left(\tilde{a}_{\boldsymbol{k}^1} d_{\boldsymbol{k}^2} + c_{\boldsymbol{k}^1} \tilde{b}_{\boldsymbol{k}^2}\right)$$

$$g_{\boldsymbol{k}} \stackrel{\text{def}}{=} Im(h_{\boldsymbol{k}}(\lambda, v)) = k_1 \tilde{L} c_{\boldsymbol{k}} + \left(\lambda + \nu k_2^4 - k_2^2\right) d_{\boldsymbol{k}} - 2k_2 \sum_{\boldsymbol{k}^1+\boldsymbol{k}^2=\boldsymbol{k}} (\tilde{a}_{\boldsymbol{k}^1} c_{\boldsymbol{k}^2} - \tilde{b}_{\boldsymbol{k}^1} d_{\boldsymbol{k}^2}).$$

Define $\mathcal{F} = \{\mathcal{F}_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathcal{I}}$ component-wise by

$$\mathcal{F}_{\boldsymbol{k}} = \begin{cases} \eta, & \boldsymbol{k} = (0,0) \\ g_{\boldsymbol{k}}, & \boldsymbol{k} = (0,k_2) \succeq (0,1) \\ \begin{pmatrix} f_{\boldsymbol{k}} \\ g_{\boldsymbol{k}} \end{pmatrix}, & \boldsymbol{k} = (k_1,k_2) \succeq (1,1). \end{cases} \tag{95}$$

Finding an eigenvalue-eigenvector pair (λ, v) satisfying (90) and $\eta = 0$ is therefore equivalent to solve

$$\mathcal{F}(x) = 0. \tag{96}$$

For sake of simplicity of the presentation, for $\boldsymbol{k} = (k_1,k_2) \succeq (0,1)$, let

$$R_{\boldsymbol{k}}(\nu,\lambda) \stackrel{\text{def}}{=} \begin{cases} \lambda + \nu k_2^4 - k_2^2, & \boldsymbol{k} = (0,k_2) \succeq (0,1) \\ \begin{pmatrix} \lambda + \nu k_2^4 - k_2^2 & -k_1 \tilde{L} \\ k_1 \tilde{L} & \lambda + \nu k_2^4 - k_2^2 \end{pmatrix}, & \boldsymbol{k} = (k_1,k_2) \succeq (1,1) \end{cases} \tag{97}$$

$$\mathcal{N}_{\boldsymbol{k}}(x) \stackrel{\text{def}}{=} \begin{cases} -2(\tilde{a} * c)_{\boldsymbol{k}} + 2(\tilde{b} * d)_{\boldsymbol{k}}, & \boldsymbol{k} = (0,k_2) \succeq (0,1) \\ \begin{pmatrix} 2(\tilde{a} * d)_{\boldsymbol{k}} + 2(\tilde{b} * c)_{\boldsymbol{k}} \\ -2(\tilde{a} * c)_{\boldsymbol{k}} + 2(\tilde{b} * d)_{\boldsymbol{k}} \end{pmatrix}, & \boldsymbol{k} = (k_1,k_2) \succeq (1,1) \end{cases} \tag{98}$$

so that for every $\boldsymbol{k} = (k_1,k_2) \succeq (0,1)$,

$$\mathcal{F}_{\boldsymbol{k}}(x) = R_{\boldsymbol{k}}(\nu,\lambda) x_{\boldsymbol{k}} + k_2 \mathcal{N}_{\boldsymbol{k}}(x). \tag{99}$$

7.1 The fixed point operator $\boldsymbol{T}$ for the eigenvalue problem

Assume an approximate solution $\bar{x} = (\bar{\lambda}, \bar{c}, \bar{d})$ of $\mathcal{F} = 0$ has been found at the parameter value ν where $\mathcal{F} = \{\mathcal{F}_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathcal{I}}$ is given component-wise by (95). Consider a *Galerkin projection* of (96) of dimension $n(\boldsymbol{m})$ given by $\mathcal{F}^{(\boldsymbol{m})} \stackrel{\text{def}}{=} \{\mathcal{F}_{\boldsymbol{k}}^{(\boldsymbol{m})}\}_{\boldsymbol{k}\in\boldsymbol{F_m}}$, where $\mathcal{F}^{(\boldsymbol{m})} \colon \mathbb{R}^{n(\boldsymbol{m})} \times \mathbb{R} \to \mathbb{R}^{n(\boldsymbol{m})}$, is given component-wise by

$$\mathcal{F}_{\boldsymbol{k}}^{(\boldsymbol{m})}(x_{\boldsymbol{F_m}}) \stackrel{\text{def}}{=} \mathcal{F}_{\boldsymbol{k}}((x_{\boldsymbol{F_m}}, 0_{\boldsymbol{I_m}})), \qquad \boldsymbol{k} \in \boldsymbol{F_m},$$

where $\mathcal{F}_{\boldsymbol{k}}((x_{\boldsymbol{F_m}}, 0_{\boldsymbol{I_m}}))$ is evaluated using (95). Now suppose that we numerically found $\bar{x}_{\boldsymbol{F_m}}$ such that $\mathcal{F}^{(\boldsymbol{m})}(\bar{x}_{\boldsymbol{F_m}}) \approx 0$. We define $\bar{x} \stackrel{\text{def}}{=} (\bar{x}_{\boldsymbol{F_m}}, 0_{\boldsymbol{I_m}}) \in X^{\boldsymbol{s}}$. Denote

$$\bar{x}_{\boldsymbol{k}} = \begin{cases} \bar{\lambda}, & \boldsymbol{k} = (0,0) \\ \bar{d}_{\boldsymbol{k}}, & \boldsymbol{k} = (0,k_2) \succeq (0,1) \\ \begin{pmatrix} \bar{c}_{\boldsymbol{k}} \\ \bar{d}_{\boldsymbol{k}} \end{pmatrix}, & \boldsymbol{k} = (k_1,k_2) \succeq (1,1), \end{cases}$$

where $\bar{c}_{\boldsymbol{k}} = \bar{d}_{\boldsymbol{k}} = 0$ for $\boldsymbol{k} \notin \boldsymbol{F_m}$. Assume the Jacobian matrix $D\mathcal{F}^{(\boldsymbol{M})}(\bar{x}_{\boldsymbol{F_m},0})$ is non-singular and let $A_{\boldsymbol{M}}$ be a numerical approximation for its inverse. To define the *tail* of the linear operator, we need the following result.

Lemma 7.4. *Let $\boldsymbol{M} = (M_1, M_2) \succeq (m_1, m_2)$, $\nu_0 > 0$ and $\bar{\lambda} \in \mathbb{R}$. If*

$$\left(\nu_0 M_2^2 - 1\right) M_2^2 + \bar{\lambda} > 0, \tag{100}$$

then $R_{\boldsymbol{k}}(\nu_0, \bar{\lambda})$ given by (97) *is invertible for all $\boldsymbol{k} \notin \boldsymbol{F_M}$.*

Proof. The proof is similar to the proof of Lemma 4.1. ☐

We define the linear operator A on sequence spaces, which acts as an approximation for the inverse of $D\mathcal{F}(\bar{x})$ as

$$\left[A(x)\right]_{\boldsymbol{k}} \stackrel{\text{def}}{=} \begin{cases} \left[A_{\boldsymbol{m}} x_{\boldsymbol{F_M}}\right]_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \in \boldsymbol{F_M} \\ R_{\boldsymbol{k}}(\nu_0, \bar{\lambda})^{-1} x_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \notin \boldsymbol{F_M}. \end{cases} \tag{101}$$

Let

$$T(x) \stackrel{\text{def}}{=} x - A\mathcal{F}(x). \tag{102}$$

Lemma 7.5. *Consider $\boldsymbol{M} = (M_1, M_2) \succeq (m_1, m_2)$ and let $\boldsymbol{s} = (s_1, s_2) \succ (1, 1)$ a decay rate. Assume that $A_{\boldsymbol{M}}$ is invertible and that* (100) *holds. Then $T : X^{\boldsymbol{s}} \to X^{\boldsymbol{s}}$ and the solutions of* (96) *are in one to one correspondence with the fixed points of T.*

Remark 7.6. Note that since we look for solutions of $\mathcal{F}(x) = 0$ given in (96) in the Banach space $X^{\boldsymbol{s}}$ given (21)

7.2 Rigorous computation of an eigenvalue-eigenvector pair

Consider bounds $Y_{\boldsymbol{k}}$ and $Z_{\boldsymbol{k}}$ for all $\boldsymbol{k} \in \mathcal{I}$, such that

$$\left| \left[T(\bar{x}) - \bar{x}\right]_{\boldsymbol{k}} \right| \le Y_{\boldsymbol{k}}, \tag{103}$$

and

$$\sup_{x_1, x_2 \in B_r(0)} \left| \left[DT(\bar{x} + x_1) x_2\right]_{\boldsymbol{k}} \right| \le Z_{\boldsymbol{k}}(r). \tag{104}$$

Lemma 7.7. *If there exists an $r > 0$ such that $\|Y + Z\|_{\boldsymbol{s}} < r$, with $Y \stackrel{\text{def}}{=} \{Y_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathcal{I}}$ and $Z \stackrel{\text{def}}{=} \{Z_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathcal{I}}$, satisfying (103) and (104), respectively, then T is a contraction mapping on $B_r(\bar{x})$ with contraction constant at most $\|Y + Z\|_{\boldsymbol{s}}/r < 1$. Furthermore, there is a unique $\tilde{x} \in B_r(\bar{x})$ such that $\mathcal{F}(\tilde{x}) = 0$, and $\tilde{x}$ lies in the interior of $B_r(\bar{x})$.*

To obtain uniform asymptotic bounds for the $Y_{\boldsymbol{k}}$, we compute $\tilde{Y}_{\boldsymbol{M}}$ such that

$$Y_{\boldsymbol{k}} = \frac{1}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \tilde{Y}_{\boldsymbol{M}} \mathbb{I}^{d(\boldsymbol{k})}, \qquad \text{for } \boldsymbol{k} \notin \boldsymbol{F_M}. \tag{105}$$

Assume there exist $\tilde{Z}_{M_1,1}(r), \dots, \tilde{Z}_{M_1,M_2-1}(r)$, and $\tilde{Z}_{\infty,M_2}(r)$ such that for $\boldsymbol{k} \notin \boldsymbol{F_M}$ with $\boldsymbol{k} = (k_1, k_2) \in I_{M_1} \times \{k_2\}$, for some $k_2 \in \{1, \dots, M_2 - 1\}$, then

$$Z_{\boldsymbol{k}}(r) = Z_{k_1,k_2}(r) \stackrel{\text{def}}{=} \frac{1}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \tilde{Z}_{M_1,k_2}(r) \mathbb{I}^2, \tag{106}$$

and for $\boldsymbol{k} \notin \boldsymbol{F_M}$ with $\boldsymbol{k} \in \mathbb{N} \times I_{M_2}$, then

$$Z_{\boldsymbol{k}}(r) = Z_{k_1,k_2}(r) \stackrel{\text{def}}{=} \frac{1}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \tilde{Z}_{\infty,M_2}(r) \mathbb{I}^{d(\boldsymbol{k})}, \tag{107}$$

where $\mathbb{I}^{d(\boldsymbol{k})} = 1$ if $d(\boldsymbol{k}) = 1$, $\mathbb{I}^{d(\boldsymbol{k})} = (1, 1)^T$ if $d(\boldsymbol{k}) = 2$.

Definition 7.8. We define the *finite radii polynomials* $\{p_{\boldsymbol{k}}(r)\}_{\boldsymbol{k}\in \boldsymbol{F_M}}$ by

$$p_{\boldsymbol{k}}(r) \stackrel{\text{def}}{=} Z_{\boldsymbol{k}}(r) - \frac{r}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}\mathbb{I}^{d(\boldsymbol{k})} + Y_{\boldsymbol{k}}, \tag{108}$$

and the *tail radii polynomials* by

$$\tilde{p}_{M_1,k_2}(r) \stackrel{\text{def}}{=} \tilde{Z}_{M_1,k_2}(r) - r + \tilde{Y}_{\boldsymbol{M}}, \qquad \text{for } k_2 = 1,\dots,M_2-1, \tag{109}$$

and

$$\tilde{p}_{\infty,M_2}(r) \stackrel{\text{def}}{=} \tilde{Z}_{\infty,M_2}(r) - r + \tilde{Y}_{\boldsymbol{M}}. \tag{110}$$

Lemma 7.9. *If there exists $r>0$ such that $p_{\boldsymbol{k}}(r)<0$ for all $\boldsymbol{k}\in\boldsymbol{F_M}$, $\tilde{p}_{M_1,k_2}(r)<0$, for all $k_2 = 1,\dots,M_2-1$ and $\tilde{p}_{\infty,M_2}(r)<0$, then there exists a unique $\tilde{x}\in B_r(\bar{x})$ such that $\mathcal{F}(\tilde{x})=0$.*

7.3 Radii polynomials for the eigenvalue problem

Recall that $(\bar{L},\bar{a},\bar{b})$ is the approximate periodic orbit and that $(\tilde{L},\tilde{a},\tilde{b})$ is the fixed point of (31) (corresponding to a periodic solution of (1) via the relation (22)) obtained from applying Lemma 5.3. Let r_γ the radius of the ball that contains the periodic orbit. Let $\hat{L} \stackrel{\text{def}}{=} \frac{1}{r_\gamma}(\tilde{L}-\bar{L})$, $\hat{a} \stackrel{\text{def}}{=} \frac{1}{r_\gamma}(\tilde{a}-\bar{a})$ and $\hat{b} \stackrel{\text{def}}{=} \frac{1}{r_\gamma}(\tilde{b}-\bar{b})$ so that $\tilde{L}=\bar{L}+r_\gamma\hat{L}$, $\tilde{a}=\bar{a}+r_\gamma\hat{a}$ and $\tilde{b}=\bar{b}+r_\gamma\hat{b}$. Then $|\hat{L}| = \frac{1}{r_\gamma}|\tilde{L}-\bar{L}|\le 1$,

$$\|\hat{a}\|_{\boldsymbol{s}} = \frac{1}{r_\gamma}\|\tilde{a}-\bar{a}\|_{\boldsymbol{s}} = \frac{1}{r_\gamma}\sup_{\boldsymbol{k}\in\mathcal{I}} \omega_{\boldsymbol{k}}^{\boldsymbol{s}}|(\tilde{a}-\bar{a})_{\boldsymbol{k}}| \le 1 \quad\text{and}\quad \|\hat{b}\|_{\boldsymbol{s}} = \frac{1}{r_\gamma}\|\tilde{b}-\bar{b}\|_{\boldsymbol{s}} = \frac{1}{r_\gamma}\sup_{\boldsymbol{k}\in\mathcal{I}} \omega_{\boldsymbol{k}}^{\boldsymbol{s}}|(\tilde{b}-\bar{b})_{\boldsymbol{k}}| \le 1.$$

Moreover,

$$\begin{aligned}
\mathcal{F}_{\boldsymbol{k}}(\bar{x}) &= R_{\boldsymbol{k}}(\nu_0,\bar{\lambda})\bar{x}_{\boldsymbol{k}} + k_2\mathcal{N}_{\boldsymbol{k}}(\bar{x}) \\
&= \begin{pmatrix} \bar{\lambda}+\nu_0k_2^4-k_2^2 & -k_1\tilde{L} \\ k_1\tilde{L} & \bar{\lambda}+\nu_0k_2^4-k_2^2 \end{pmatrix}\begin{pmatrix} \bar{c}_{\boldsymbol{k}} \\ \bar{d}_{\boldsymbol{k}} \end{pmatrix} + k_2\begin{pmatrix} 2(\tilde{a}*\bar{d})_{\boldsymbol{k}} + 2(\tilde{b}*\bar{c})_{\boldsymbol{k}} \\ -2(\tilde{a}*\bar{c})_{\boldsymbol{k}} + 2(\tilde{b}*\bar{d})_{\boldsymbol{k}} \end{pmatrix} \\
&= \begin{pmatrix} \bar{\lambda}+\nu_0k_2^4-k_2^2 & -k_1\bar{L} \\ k_1\bar{L} & \bar{\lambda}+\nu_0k_2^4-k_2^2 \end{pmatrix}\begin{pmatrix} \bar{c}_{\boldsymbol{k}} \\ \bar{d}_{\boldsymbol{k}} \end{pmatrix} + k_2\begin{pmatrix} 2(\bar{a}*\bar{d})_{\boldsymbol{k}} + 2(\bar{b}*\bar{c})_{\boldsymbol{k}} \\ -2(\bar{a}*\bar{c})_{\boldsymbol{k}} + 2(\bar{b}*\bar{d})_{\boldsymbol{k}} \end{pmatrix} \\
&\quad + \left[\begin{pmatrix} 0 & -k_1\hat{L} \\ k_1\hat{L} & 0 \end{pmatrix}\begin{pmatrix} \bar{c}_{\boldsymbol{k}} \\ \bar{d}_{\boldsymbol{k}} \end{pmatrix} + k_2\begin{pmatrix} 2(\hat{a}*\bar{d})_{\boldsymbol{k}} + 2(\hat{b}*\bar{c})_{\boldsymbol{k}} \\ -2(\hat{a}*\bar{c})_{\boldsymbol{k}} + 2(\hat{b}*\bar{d})_{\boldsymbol{k}} \end{pmatrix}\right] r_\gamma.
\end{aligned}$$

For all $\boldsymbol{k}\succeq(1,1)$, denote

$$\bar{\mathcal{F}}_{\boldsymbol{k}}(\bar{x}) \stackrel{\text{def}}{=} \begin{pmatrix} \bar{\lambda}+\nu_0k_2^4-k_2^2 & -k_1\bar{L} \\ k_1\bar{L} & \bar{\lambda}+\nu_0k_2^4-k_2^2 \end{pmatrix}\begin{pmatrix} \bar{c}_{\boldsymbol{k}} \\ \bar{d}_{\boldsymbol{k}} \end{pmatrix} + k_2\begin{pmatrix} 2(\bar{a}*\bar{d})_{\boldsymbol{k}} + 2(\bar{b}*\bar{c})_{\boldsymbol{k}} \\ -2(\bar{a}*\bar{c})_{\boldsymbol{k}} + 2(\bar{b}*\bar{d})_{\boldsymbol{k}} \end{pmatrix} \tag{111}$$

Hence,

$$|\mathcal{F}_{\boldsymbol{k}}(\bar{x})| \le \left|\bar{\mathcal{F}}_{\boldsymbol{k}}(\bar{x})\right| + \rho_{\boldsymbol{k}}r_\gamma \stackrel{\text{def}}{=} \left|\bar{\mathcal{F}}_{\boldsymbol{k}}(\bar{x})\right| + \begin{pmatrix} k_1|\bar{d}_{\boldsymbol{k}}| + 4k_2(\|\bar{c}\|_{\boldsymbol{s}} + \|\bar{d}\|_{\boldsymbol{s}})\alpha_{k_1,s_1}^{(0,1)}\alpha_{k_2,s_2}^{(0,0)} \\ k_1|\bar{c}_{\boldsymbol{k}}| + 2k_2\|\bar{c}\|_{\boldsymbol{s}}\alpha_{k_1,s_1}^{(0,0)}\alpha_{k_2,s_2}^{(0,0)} + 2k_2\|\bar{d}\|_{\boldsymbol{s}}\alpha_{k_1,s_1}^{(1,1)}\alpha_{k_2,s_2}^{(0,0)} \end{pmatrix} r_\gamma.$$

Since $T(\bar{x})-\bar{x} = -A\mathcal{F}(\bar{x})$, let

$$Y_{\boldsymbol{k}} \stackrel{\text{def}}{=} \left|\left[A_{\boldsymbol{M}}\bar{\mathcal{F}}^{(\boldsymbol{M})}(\bar{x}_{\boldsymbol{F_m}},0)\right]_{\boldsymbol{k}}\right| + |[A_{\boldsymbol{M}}\rho_{\boldsymbol{F_M}}]_{\boldsymbol{k}}|\, r_\gamma, \qquad \boldsymbol{k}\in\boldsymbol{F_M}. \tag{112}$$

To compute the uniform asymptotic bounds $\tilde{Y}_{\boldsymbol{M}}$ satisfying (105), notice that $\bar{\mathcal{F}}_{\boldsymbol{k}}(\bar{x})=0$ for every $\boldsymbol{k}\notin\boldsymbol{F_{\bar{M}}}$, with $\boldsymbol{F_{\bar{M}}}$ given by (48). Hence, set

$$\begin{aligned}
\tilde{Y}_{\boldsymbol{M}} \stackrel{\text{def}}{=} &\max_{\boldsymbol{k}\in\boldsymbol{F_{\bar{M}}}\setminus\boldsymbol{F_M}} \left|R_{\boldsymbol{k}}(\nu_0,\bar{\lambda})^{-1}\bar{\mathcal{F}}_{\boldsymbol{k}}(\bar{x})\right|\omega_{\boldsymbol{k}}^{\boldsymbol{s}} \\
&+ 2\left\|\begin{pmatrix} \tilde{\beta}_{\infty,M_2}^{(1)} & \tilde{\beta}_{\infty,M_2}^{(2)} \\ \tilde{\beta}_{\infty,M_2}^{(2)} & \tilde{\beta}_{\infty,M_2}^{(1)} \end{pmatrix}\begin{pmatrix} 2(\|\bar{c}\|_{\boldsymbol{s}} + \|\bar{d}\|_{\boldsymbol{s}})\alpha_{M_1,s_1}^{(0,1)}\alpha_{M_2,s_2}^{(0,0)} \\ \|\bar{c}\|_{\boldsymbol{s}}\alpha_{M_1,s_1}^{(0,0)}\alpha_{M_2,s_2}^{(0,0)} + \|\bar{d}\|_{\boldsymbol{s}}\alpha_{M_1,s_1}^{(1,1)}\alpha_{M_2,s_2}^{(0,0)} \end{pmatrix}\right\|_\infty r_\gamma.
\end{aligned} \tag{113}$$

Remark 7.10. If $\boldsymbol{M} = (M_1, M_2) \succeq \bar{\boldsymbol{M}} = (2m_1 - 1, 2m_2 - 1)$, then the set $\boldsymbol{F}_{\bar{\boldsymbol{M}}} \setminus \boldsymbol{F}_{\boldsymbol{M}}$ is empty. In this case, we let $\tilde{Y}_{\boldsymbol{M}} = 0$.

In order to compute the bound $Z_{\boldsymbol{k}}$ satisfying (104), for all $\boldsymbol{k}$, it is convenient to introduce the operator $A^\dagger$ whose action on a vector $x \in X^{\boldsymbol{s}}$ is given component-wise by

$$\left[A^\dagger(x)\right]_{\boldsymbol{k}} \stackrel{\text{def}}{=} \begin{cases} \left[D\mathcal{F}^{(\boldsymbol{M})}(\bar{x}_{\boldsymbol{F}_m}, 0) x_{\boldsymbol{F}_{\boldsymbol{M}}}\right]_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \in \boldsymbol{F}_{\boldsymbol{M}} \\ R_{\boldsymbol{k}}(\nu_0, \bar{\lambda}) x_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \notin \boldsymbol{F}_{\boldsymbol{M}}, \end{cases} \tag{114}$$

which acts as an approximate inverse for the operator A defined in (101). We consider the splitting

$$DT(\bar{x} + x_1)x_2 = \left(I - AA^\dagger\right)x_2 - A\left(D\mathcal{F}(\bar{x} + x_1) - A^\dagger\right)x_2, \tag{115}$$

where the first term is small for $\boldsymbol{k} \in \boldsymbol{F}_{\boldsymbol{M}}$, and is zero for $\boldsymbol{k} \notin \boldsymbol{F}_{\boldsymbol{M}}$. For $\boldsymbol{k} \in \boldsymbol{F}_{\boldsymbol{M}}$, we have that

$$\left|\left[(I - AA^\dagger)x_2\right]_{\boldsymbol{k}}\right| \leq \left[\left|I - A_{\boldsymbol{M}} D\mathcal{F}^{(\boldsymbol{M})}(\bar{x}_{\boldsymbol{F}_m}, 0)\right| \omega^{-\boldsymbol{s}}_{\boldsymbol{F}_{\boldsymbol{M}}}\right]_{\boldsymbol{k}} r \tag{116}$$

where $\omega^{-\boldsymbol{s}}_{\boldsymbol{F}_{\boldsymbol{M}}} \stackrel{\text{def}}{=} \{\omega_{\boldsymbol{k}}^{-\boldsymbol{s}} \mathbb{I}^{d(\boldsymbol{k})}\}_{\boldsymbol{k} \in \boldsymbol{F}_{\boldsymbol{M}}}$, and $|\cdot|$ represents component-wise absolute values. Consider $u, v \in B_1(0)$ defined by $x_1 = ru$ and $x_2 = rv$ so that we can expand the expression $\left[\left(D\mathcal{F}(\bar{x} + x_1) - A^\dagger\right)x_2\right]_{\boldsymbol{k}}$ in terms of r. Denote $u = (\lambda^{(u)}, c^{(u)}, d^{(u)})$ and $v = (\lambda^{(v)}, c^{(v)}, d^{(v)})$ component-wise as in (54). We have that $\left[\left(D\mathcal{F}(\bar{x} + ru) - A^\dagger\right)rv\right]_{0,0} = 0$.

For $\boldsymbol{k} \in \mathcal{I} \setminus \{(0,0)\}$, consider the expansion

$$\left[\left(D\mathcal{F}(\bar{x} + ru) - A^\dagger\right) rv\right]_{\boldsymbol{k}} = C^{(2)}_{\boldsymbol{k}} r^2 + C^{(1)}_{\boldsymbol{k}} r,$$

where the coefficients are given by

$$C^{(1)}_{\boldsymbol{k}} \stackrel{\text{def}}{=} \begin{cases} 2k_2 \left[\begin{pmatrix} (\bar{a} * d^{(v)}_{\boldsymbol{I_M}})_{\boldsymbol{k}} + (c^{(v)}_{\boldsymbol{I_M}} * \bar{b})_{\boldsymbol{k}} \\ -(\bar{a} * c^{(v)}_{\boldsymbol{I_M}})_{\boldsymbol{k}} + (\bar{b} * d^{(v)}_{\boldsymbol{I_M}})_{\boldsymbol{k}} \end{pmatrix} + \begin{pmatrix} (\hat{a} * d^{(v)}_{\boldsymbol{I_M}})_{\boldsymbol{k}} + (c^{(v)}_{\boldsymbol{I_M}} * \hat{b})_{\boldsymbol{k}} \\ -(\hat{a} * c^{(v)}_{\boldsymbol{I_M}})_{\boldsymbol{k}} + (\hat{b} * d^{(v)}_{\boldsymbol{I_M}})_{\boldsymbol{k}} \end{pmatrix} r_\gamma\right], & \text{if } \boldsymbol{k} \in \boldsymbol{F}_{\boldsymbol{M}} \\ 2k_2 \left[\begin{pmatrix} (\bar{a} * d^{(v)})_{\boldsymbol{k}} + (c^{(v)} * \bar{b})_{\boldsymbol{k}} \\ -(\bar{a} * c^{(v)})_{\boldsymbol{k}} + (\bar{b} * d^{(v)})_{\boldsymbol{k}} \end{pmatrix} + \begin{pmatrix} (\hat{a} * d^{(v)})_{\boldsymbol{k}} + (c^{(v)} * \hat{b})_{\boldsymbol{k}} \\ -(\hat{a} * c^{(v)})_{\boldsymbol{k}} + (\hat{b} * d^{(v)})_{\boldsymbol{k}} \end{pmatrix} r_\gamma\right], & \text{if } \boldsymbol{k} \notin \boldsymbol{F}_{\boldsymbol{M}}, \end{cases}$$

$$C^{(2)}_{\boldsymbol{k}} \stackrel{\text{def}}{=} \begin{pmatrix} \lambda^{(v)} c^{(u)}_{\boldsymbol{k}} + \lambda^{(u)} c^{(v)}_{\boldsymbol{k}} \\ \lambda^{(v)} d^{(u)}_{\boldsymbol{k}} + \lambda^{(u)} d^{(v)}_{\boldsymbol{k}} \end{pmatrix}.$$

For any k_2, the first components of $C^{(1,0)}_{0,k_2}$ and $C^{(2,0)}_{0,k_2}$ equal 0. We are ready to compute the bounds $Z_k(r)$ satisfying (104). In Section 7.3.1, we compute the bounds for $\boldsymbol{k} \in \boldsymbol{F}_{\boldsymbol{M}}$, and in Section 7.3.2, we compute the bounds for $\boldsymbol{k} \notin \boldsymbol{F}_{\boldsymbol{M}}$.

7.3.1 The bound $Z_k(r)$, for $k \in F_M$

In order to compute the bounds $Z_k(r)$ for $\boldsymbol{k} \in \boldsymbol{F}_{\boldsymbol{M}}$, we introduce intermediate upper bounds $z^{(1,0)}_{\boldsymbol{k}}, z^{(2,0)}_{\boldsymbol{k}}$ such that $|C^{(1,0)}_{\boldsymbol{k}}| \leq z^{(1,0)}_{\boldsymbol{k}}$ and $|C^{(2,0)}_{\boldsymbol{k}}| \leq z^{(2,0)}_{\boldsymbol{k}}$. These are given by

$$z^{(1,0)}_{\boldsymbol{k}} \stackrel{\text{def}}{=} 2|k_2| \left[\begin{pmatrix} (|\bar{a}| * \omega^{(-\boldsymbol{s},b)}_{\boldsymbol{I_M}})_{\boldsymbol{k}} + (\omega^{(-\boldsymbol{s},a)}_{\boldsymbol{I_M}} * |\bar{b}|)_{\boldsymbol{k}} \\ (|\bar{a}| * \omega^{(-\boldsymbol{s},a)}_{\boldsymbol{I_M}})_{\boldsymbol{k}} + (|\bar{b}| * \omega^{(-\boldsymbol{s},b)}_{\boldsymbol{I_M}})_{\boldsymbol{k}} \end{pmatrix} + \begin{pmatrix} 2\alpha^{(0,1)}_{k_1,s_1} \alpha^{(0,0)}_{k_2,s_2} \\ \alpha^{(0,0)}_{k_1,s_1} \alpha^{(0,0)}_{k_2,s_2} + \alpha^{(1,1)}_{k_1,s_1} \alpha^{(0,0)}_{k_2,s_2} \end{pmatrix} r_\gamma \omega^{-\boldsymbol{s}}_{\boldsymbol{k}}\right]$$

$$z^{(2,0)}_{\boldsymbol{k}} \stackrel{\text{def}}{=} 2 \begin{pmatrix} \omega^{(-\boldsymbol{s},b)}_{\boldsymbol{k}} \\ \omega^{(-\boldsymbol{s},a)}_{\boldsymbol{k}} \end{pmatrix}.$$

Letting

$$Z_{\boldsymbol{k}}^{(1)} \stackrel{\text{def}}{=} \left[\left| I - A_{\boldsymbol{M}} D\mathcal{F}^{(\boldsymbol{M})}(\bar{x}) \right| \omega_{\boldsymbol{F_M}}^{-\boldsymbol{s}} \right]_{\boldsymbol{k}} + \left[|A_{\boldsymbol{M}}| \, z_{\boldsymbol{F_M}}^{(1,0)} \right]_{\boldsymbol{k}}$$
$$Z_{\boldsymbol{k}}^{(2)} \stackrel{\text{def}}{=} \left(|A_{\boldsymbol{M}}| \, z_{\boldsymbol{F_M}}^{(2,0)} \right)_{\boldsymbol{k}}$$

we set, for $\boldsymbol{k} \in \boldsymbol{F_M}$,

$$Z_k(r) \stackrel{\text{def}}{=} Z_{\boldsymbol{k}}^{(2)} r^2 + Z_{\boldsymbol{k}}^{(1)} r. \tag{117}$$

7.3.2 The bound $Z_{\boldsymbol{k}}(r, |\boldsymbol{\Delta_\nu}|)$ for $\boldsymbol{k} \notin \boldsymbol{F_M}$

Consider fixed $\bar{\lambda} > 0$ and $\nu_0 > 0$. For any $\boldsymbol{k} \notin \boldsymbol{F_M}$, let

$$\beta_{\boldsymbol{k}}^{(1)} \stackrel{\text{def}}{=} \frac{|\bar{\lambda} + \nu_0 k_2^4 - k_2^2| k_2}{(\bar{\lambda} + \nu_0 k_2^4 - k_2^2)^2 + (k_1 \tilde{L})^2} \qquad \text{and} \qquad \beta_{\boldsymbol{k}}^{(2)} \stackrel{\text{def}}{=} \frac{k_1 k_2 \tilde{L}}{(\bar{\lambda} + \nu_0 k_2^4 - k_2^2)^2 + (k_1 \tilde{L})^2}.$$

Lemma 7.11. *Consider $\bar{L} > 0$ and $\nu > 0$. Given $\boldsymbol{M} = (M_1, M_2)$, consider*

$$k_2^* \stackrel{\text{def}}{=} \max_{k_2 \in \{1,\dots,M_2-1\}} \left\{ k_2 \;\middle|\; |\bar{\lambda} + \nu_0 k_2^4 - k_2^2| \le M_1 \tilde{L} \right\}. \tag{118}$$

Assume that

$$\nu_0 > \frac{1}{(k_2^* + 1)^2}. \tag{119}$$

For $k_2 = 1, \dots, M_2 - 1$, let

$$\tilde{\beta}_{M_1,k_2}^{(1)} \stackrel{\text{def}}{=} \frac{|\bar{\lambda} + \nu_0 k_2^4 - k_2^2| k_2}{(\bar{\lambda} + \nu_0 k_2^4 - k_2^2)^2 + (M_1 \tilde{L})^2} \tag{120}$$

$$\tilde{\beta}_{M_1,k_2}^{(2)} \stackrel{\text{def}}{=} \begin{cases} \dfrac{k_2 M_1 \tilde{L}}{(\bar{\lambda} + \nu_0 k_2^4 - k_2^2)^2 + (M_1 \tilde{L})^2}, & k_2 = 1, \dots, k_2^* \\ \dfrac{1}{2(\nu_0 k_2^3 - 1 - \frac{|\bar{\lambda}|}{k_2})}, & k_2 = k_2^* + 1, \dots, M_2 - 1 \end{cases} \tag{121}$$

and let

$$\tilde{\beta}_{\infty,M_2}^{(1)} \stackrel{\text{def}}{=} \frac{1}{\nu_0 M_2^3 - 1 - \frac{|\bar{\lambda}|}{M_2}} \quad \textit{and} \quad \tilde{\beta}_{\infty,M_2}^{(2)} \stackrel{\text{def}}{=} \frac{1}{2(\nu_0 M_2^3 - 1 - \frac{|\bar{\lambda}|}{M_2})}.$$

Let $\boldsymbol{k} \notin \boldsymbol{F_M}$. If $\boldsymbol{k} \in I_{M_1} \times \{k_2\}$ for some $k_2 = 1, \dots, M_2 - 1$, then $\beta_{\boldsymbol{k}}^{(1)}(\nu) \le \tilde{\beta}_{M_1,k_2}^{(1)}$ and $\beta_{\boldsymbol{k}}^{(2)}(\nu) \le \tilde{\beta}_{M_1,k_2}^{(1)}$. If $\boldsymbol{k} \in \mathbb{N} \times I_{M_2}$, then $\beta_{\boldsymbol{k}}^{(1)}(\nu) \le \tilde{\beta}_{\infty,M_2}^{(1)}$ and $\beta_{\boldsymbol{k}}^{(2)}(\nu) \le \tilde{\beta}_{\infty,M_2}^{(2)}$.

Proof. The proof is similar to the proof of Lemma 6.5. □

Lemma 7.12. *Let*

$$V_{\infty,M_2}^{(1)} \stackrel{\text{def}}{=} \left\| 2 \begin{pmatrix} \tilde{\beta}_{\infty,M_2}^{(1)} & \tilde{\beta}_{\infty,M_2}^{(2)} \\ \tilde{\beta}_{\infty,M_2}^{(2)} & \tilde{\beta}_{\infty,M_2}^{(1)} \end{pmatrix} \begin{pmatrix} (\|\bar{a}\|_{\boldsymbol{s}} + \|\bar{b}\|_{\boldsymbol{s}} + 2r_\gamma) \alpha_{M_1,s_1}^{(0,1)} \alpha_{M_2,s_2}^{(0,0)} \\ (\|\bar{a}\|_{\boldsymbol{s}} + r_\gamma) \alpha_{M_1,s_1}^{(0,0)} \alpha_{M_2,s_2}^{(0,0)} + (\|\bar{b}\|_{\boldsymbol{s}} + r_\gamma) \alpha_{M_1,s_1}^{(1,1)} \alpha_{M_2,s_2}^{(0,0)} \end{pmatrix} \right\|_\infty \tag{122}$$

$$V_{\infty,M_2}^{(2)} \stackrel{\text{def}}{=} 2 \left(\tilde{\beta}_{\infty,M_2}^{(1)} + \tilde{\beta}_{\infty,M_2}^{(2)} \right). \tag{123}$$

Consider $\boldsymbol{k} \notin \boldsymbol{F_M}$ with $\boldsymbol{k} \in \mathbb{N} \times I_{M_2}$. Then, for $i = 1, 2$,

$$\|R_{\boldsymbol{k}}(\nu_0, \bar{\lambda})^{-1} C_{\boldsymbol{k}}^{(i)}\|_\infty \le V_{\infty,M_2}^{(i)} \omega_{\boldsymbol{k}}^{-\boldsymbol{s}}. \tag{124}$$

Defining the bound

$$\tilde{Z}_{\infty,M_2}(r) \stackrel{\text{def}}{=} V^{(2)}_{\infty,M_2} r^2 + V^{(1)}_{\infty,M_2} r, \tag{125}$$

we get that for any $\boldsymbol{k} \notin \boldsymbol{F_M}$ with $\boldsymbol{k} \in \mathbb{N} \times I_{M_2}$

$$\sup_{x_1,x_2 \in B_r(0)} \|[DT(\bar{x}+x_1)x_2]_{\boldsymbol{k}}\|_\infty \le \sup_{x_1,x_2 \in B_r(0)} \left\| R_{\boldsymbol{k}}(\nu_0,\bar{\lambda})^{-1} \left(C^{(1)}_{\boldsymbol{k}} r + C^{(2)}_{\boldsymbol{k}} r^2 \right) \right\|_\infty \le \tilde{Z}_{\infty,M_2}(r)\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}.$$

Lemma 7.13. *Let $\boldsymbol{k} \notin \boldsymbol{F_M}$ with $\boldsymbol{k} \in I_{M_1} \times \{k_2\}$, for some $k_2 = 1, \ldots, M_2 - 1$ and let*

$$V^{(1)}_{M_1,k_2} \stackrel{\text{def}}{=} \left\| 2 \begin{pmatrix} \tilde{\beta}^{(1)}_{M_1,k_2} & \tilde{\beta}^{(2)}_{M_1,k_2} \\ \tilde{\beta}^{(2)}_{M_1,k_2} & \tilde{\beta}^{(1)}_{M_1,k_2} \end{pmatrix} \begin{pmatrix} \Psi_{M_1,k_2}(\bar{a}) + \Psi_{M_1,k_2}(\bar{b}) + 2\alpha^{(0,1)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2} r_\gamma \\ \Psi_{M_1,k_2}(\bar{a}) + \Psi_{M_1,k_2}(\bar{b}) + (\alpha^{(0,0)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2} + \alpha^{(1,1)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2}) r_\gamma \end{pmatrix} \right\|_\infty \tag{126}$$

$$V^{(2)}_{M_1,k_2} \stackrel{\text{def}}{=} 2. \tag{127}$$

Then, for $i = 1, 2$,

$$\|R_{\boldsymbol{k}}(\nu_0,\bar{\lambda})^{-1} C^{(i)}_{\boldsymbol{k}}\|_\infty \le V^{(i)}_{M_1,k_2}\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}.$$

For each $k_2 = 1, \ldots, M_2 - 1$, defining the bound

$$\tilde{Z}_{M_1,k_2}(r) \stackrel{\text{def}}{=} V^{(2)}_{M_1,k_2} r^2 + V^{(1)}_{M_1,k_2} r, \tag{128}$$

we get that for any $\boldsymbol{k} \notin \boldsymbol{F_M}$ with $\boldsymbol{k} \in I_{M_1} \times \{k_2\}$, for some $k_2 = 1, \ldots, M_2 - 1$

$$\sup_{x_1,x_2 \in B_r(0)} \|[DT(\bar{x}+x_1)x_2]_{\boldsymbol{k}}\|_\infty \le \tilde{Z}_{M_1,k_2}(r)\omega_{\boldsymbol{k}}^{-\boldsymbol{s}}.$$

The radii polynomials of Definition 7.8 can now be defined.
Combining (112) and (117), we set

$$p_{\boldsymbol{k}}(r) \stackrel{\text{def}}{=} Z^{(2)}_{\boldsymbol{k}} r^2 + \left(Z^{(1)}_{\boldsymbol{k}} - \omega_{\boldsymbol{k}}^{-\boldsymbol{s}} \mathbb{I}^{d(\boldsymbol{k})} \right) r + Y_{\boldsymbol{k}}, \qquad \boldsymbol{k} \in \boldsymbol{F_M}. \tag{129}$$

Using (113) and (125), let

$$\tilde{p}_{\infty,M_2}(r) = \tilde{Z}_{\infty,M_2}(r) - r + \tilde{Y}_{\boldsymbol{M}}, \tag{130}$$

and using (128), for $k_2 = 1, \ldots, M_2 - 1$, let

$$\tilde{p}_{M_1,k_2}(r) \stackrel{\text{def}}{=} \tilde{Z}_{M_1,k_2}(r) - r + \tilde{Y}_{\boldsymbol{M}}. \tag{131}$$

8 Results

In this final section, we present several sample theorems about existence of periodic orbits. Given a numerical solution $\bar{x} = (\bar{L}, \bar{a}, \bar{b}) \in \mathbb{R}^{n(\boldsymbol{m})}$ with $n(\boldsymbol{m}) = 2m_1m_2 - 2m_1 - m_2 + 2$ and a parameter value ν, denote the associated approximate periodic solution by

$$\bar{u}(t,y) = \sum_{\boldsymbol{k} \in \boldsymbol{F_m}} (\bar{a}_{\boldsymbol{k}} + i\bar{b}_{\boldsymbol{k}}) e^{i\bar{L}k_1 t} e^{ik_2 y}. \tag{132}$$

Also, given an exact solution $\tilde{x} = (\tilde{L}, \tilde{a}, \tilde{b}) \in B_r(\bar{x})$ of $\mathcal{F}(\tilde{x}, \nu) = 0$, denote the associated exact periodic solution by

$$\tilde{u}(t,y) = \sum_{\boldsymbol{k} \in \mathbb{Z}^2} (\tilde{a}_{\boldsymbol{k}} + i\tilde{b}_{\boldsymbol{k}}) e^{i\tilde{L}k_1 t} e^{ik_2 y}. \tag{133}$$

8.1 Global branch of periodic orbits

The proof of the following result is obtained by applying Algorithm 6.9.

Theorem 8.1. *For each segment of curve in the diagram of Figure 2 there exists a smooth branch of periodic orbits of* (1)*. The branch is parameterized by* $\nu \in [\nu_{min}, \nu_{max}] = [0.12175, 0.1310]$*, and the period of the orbits ranges in the interval* $\tau \in [\tau_{min}, \tau_{max}] = [1.92610, 3.03565]$.

In Table 1, we provide data for the periodic orbit of Theorem 8.1 at the parameter value $\nu = 0.127$. This orbit was also proven with $\Delta_\nu = 2{\times}10^{-5}$. Note that in [18], the proof of the same orbit was performed with $\Delta_\nu \approx 10^{-7}$. The branch of periodic orbits of Theorem 8.1 parameterized over $\nu \in [\nu_{\min}, \nu_{\max}]$ is portrayed in Figure 2.

$\boldsymbol{m} = (m_1, m_2)$	$\boldsymbol{M} = (M_1, M_2)$	Time (in seconds)	Δ_ν	C^0 and L^2 errors
$\boldsymbol{m} = (9, 14)$	$\boldsymbol{M} = (63, 40)$	51.64631	0	$\approx 10^{-2}$
$\boldsymbol{m} = (30, 15)$	$\boldsymbol{M} = (64, 40)$	54.8595	2×10^{-5}	$\approx 10^{-3}$
$\boldsymbol{m} = (40, 80)$	$\boldsymbol{M} = (64, 80)$	460.50071	2×10^{-5}	$\approx 10^{-11}$

Table 1: Proofs data of the periodic of Theorem 8.1 at $\nu = 0.127$ with period $\tau \approx 2.2443335614892281$.

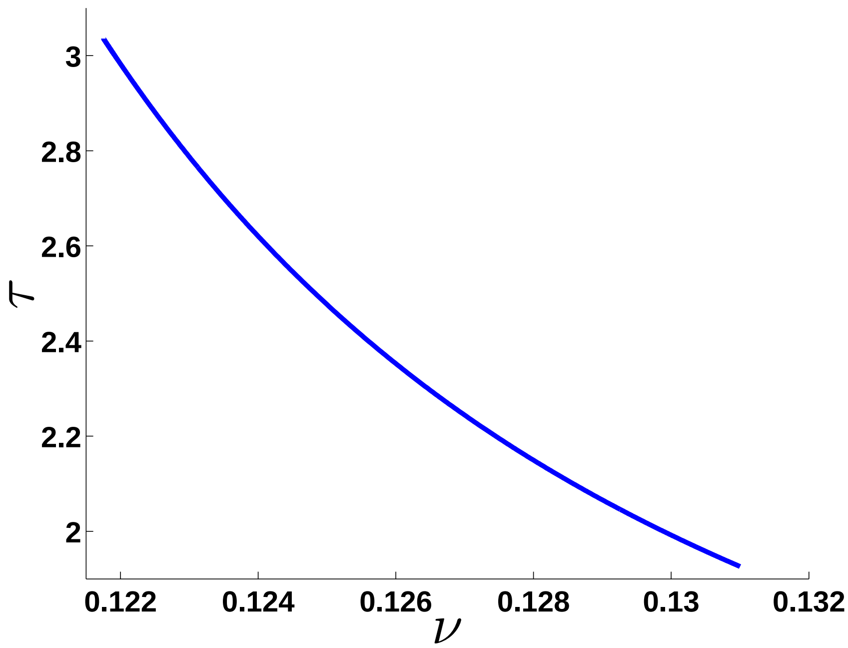

Figure 2: Theorem 8.1 demonstrates the existence of a smooth branch of periodic orbits of the Kuramoto-Sivashinsky PDE (1). For each $\nu \in [\nu_{\min}, \nu_{\max}] = [0.12175, 0.1310]$, each true solution $\tilde{x}_\nu$ is proved to lie in a ball, given by (33), of radius $r = 7.98829 \times 10^{-5}$ around the numerical approximation $\bar{x}_\nu$. More explicitly, the true solution $\tilde{x}_\nu$ is such that $\|\tilde{x}_\nu - \bar{x}_\nu\|_{\boldsymbol{s}} \le r$, with the decay rate $\boldsymbol{s} = (s_1, s_2) = (1.4, 1.2)$.

8.2 Sample theorems at fixed parameter values

Finally, we present several sample theorems at fixed parameter values. Each existence proof is obtained by fixing $\Delta_\nu = 0$, ny constructing the radii polynomials defined in (79), (80) and (81), and then by applying Lemma 5.3 successfully. Each result concerning Floquet exponents is obtained by constructing the radii polynomials (129), (130) and (131), and by applying Lemma 7.9.

Theorem 8.2. *Let $\nu = 0.032$, $\boldsymbol{m} = (m_1, m_2) = (40, 80)$, $\boldsymbol{M} = (M_1, M_2) = (200, 82)$ and $\boldsymbol{s} = (s_1, s_2) = (1.4, 1.2)$. Consider $\bar{x} = (\bar{L}, \bar{a}, \bar{b}) \in \mathbb{R}^{n(\boldsymbol{m})} = \mathbb{R}^{6242}$ given in the file* `x0pt032.txt`*, and let $\bar{u}(t, y)$ the associate approximation given by (132). Then there exists a function $\tilde{u}(t, y)$, a classical solution of (1) such that $\tilde{u}$ is periodic with respect to t with period $\tau \in 0.818577285749405 + \delta[-1, 1]$, with $\delta = 1.061985 \times 10^{-13}$, and such that*

$$\|\tilde{u}(0, \cdot) - \bar{u}(0, \cdot)\|_{C^0} \leq 1.74978 \times 10^{-10}, \qquad \text{and} \qquad \|\tilde{u}(0, \cdot) - \bar{u}(0, \cdot)\|_{L^2} \leq 2.78737 \times 10^{-11}.$$

Note that the periodic orbit of Theorem 8.2 is apparently stable, but we do not have a proof of this statement. Note also that in [18], the proof of the same orbit was performed with $m_2 = 23$ space Fourier modes, the C^0 error norm was 9.46×10^{-4}, the L^2 error norm was 9.59×10^{-4} and the error norm for the period was 4×10^{-4} (compare with 1.07×10^{-13} with our approach).

Theorem 8.3. *Let $\nu = 0.029909$, $\boldsymbol{m} = (m_1, m_2) = (50, 80)$, $\boldsymbol{M} = (M_1, M_2) = (240, 85)$ and $\boldsymbol{s} = (s_1, s_2) = (1.4, 1.2)$. Consider $\bar{x} = (\bar{L}, \bar{a}, \bar{b}) \in \mathbb{R}^{7822}$ given in the file* `x0pt02991.txt` *and consider the associate approximation $\bar{u}(t, y)$ given in (132). Then there exists a function $\tilde{u}(t, y)$, a classical solution of (1) such that $\tilde{u}$ is periodic with respect to t with period $\tau \in 0.898089445890309 + \delta[-1, 1]$, with $\delta = 1.77012 \times 10^{-13}$, and such that*

$$\|\tilde{u}(0, \cdot) - \bar{u}(0, \cdot)\|_{C^0} \leq 2.42296 \times 10^{-10}, \qquad \text{and} \qquad \|\tilde{u}(0, \cdot) - \bar{u}(0, \cdot)\|_{L^2} \leq 3.85972 \times 10^{-11}.$$

Moreover, there exists a positive real Floquet exponent $\lambda > 0$ (in fact it is a positive Lyapunov exponent) with $\lambda \approx 1.337449912731968$ associated to the periodic orbit. Since $|e^{\lambda \tau}| > 1$, then the periodic orbit is unstable. The periodic orbit is portrayed in Figure 3.

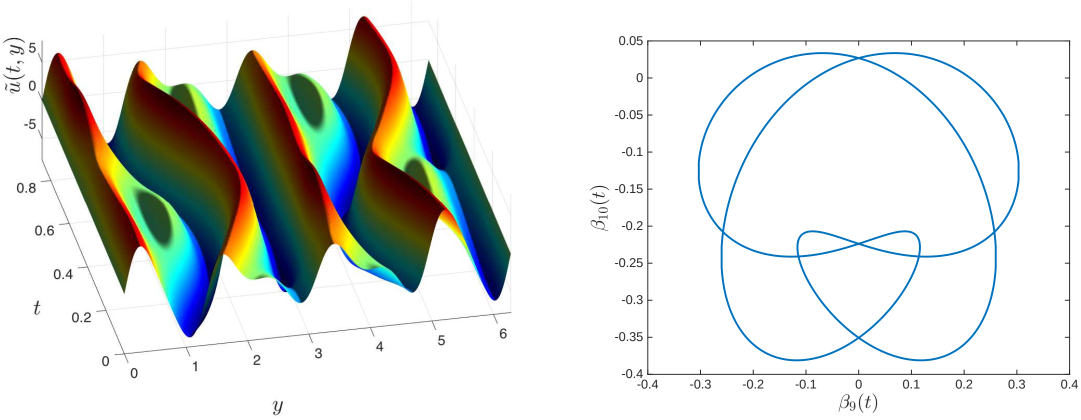

Figure 3: **(Left)** The periodic orbit of Theorem 8.3 at $\nu = 0.02991$. **(Right)** The Fourier modes $(\beta_9(t), \beta_{10}(t))$, where the $\beta_{k_2}(t)$ are given by (139).

Theorem 8.4. *Let $\nu = 4/150$, $\boldsymbol{m} = (m_1, m_2) = (60, 90)$, $\boldsymbol{M} = (M_1, M_2) = (180, 90)$ and $\boldsymbol{s} = (s_1, s_2) = (1.4, 1.2)$. Consider $\bar{x} = (\bar{L}, \bar{a}, \bar{b}) \in \mathbb{R}^{10592}$ given in the file* `x_4over450.txt` *and consider the associate approximation $\bar{u}(t, y)$ given in (132). Then there exists a function $\tilde{u}(t, y)$, a classical solution of (1) such that $\tilde{u}$ is periodic with respect to t with period $\tau \in 0.534173514309819 + \delta[-1, 1]$, with $\delta = 9.925582 \times 10^{-14}$, and such that*

$$\|\tilde{u}(0, \cdot) - \bar{u}(0, \cdot)\|_{C^0} \leq 3.84039 \times 10^{-10}, \qquad \text{and} \qquad \|\tilde{u}(0, \cdot) - \bar{u}(0, \cdot)\|_{L^2} \leq 6.11767 \times 10^{-11}.$$

Let $p = 2\tau$. Then there exists an eigenvalue-eigenvector pair (λ, v) of (90) with $v(t + p, y) = v(t, y)$ for all $t \in \mathbb{R}$ and for all $y \in [0, 2\pi]$ such that $\lambda > 0$ with $\lambda \approx 1.018412816297953$. Since $|e^{\lambda \tau}| > 1$, then the periodic orbit is unstable. The periodic orbit is portrayed in Figure 4.

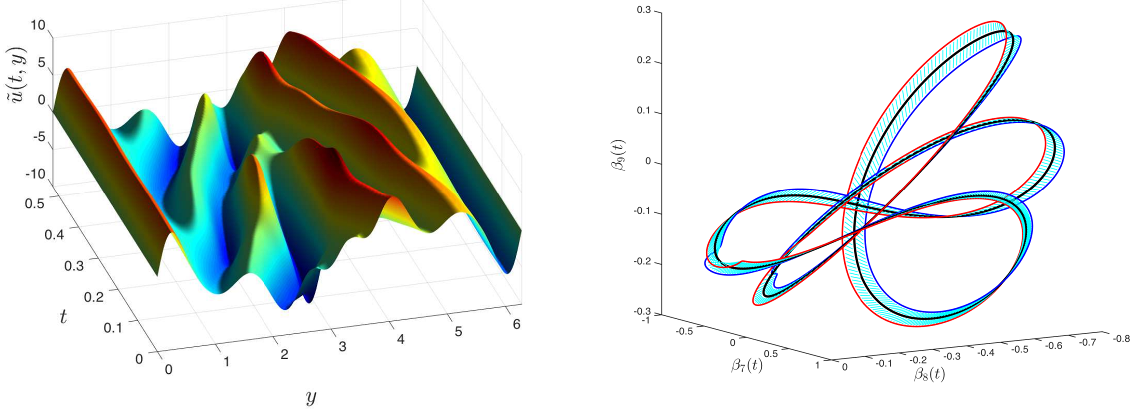

Figure 4: **(Left)** The profile of a periodic orbit of (1) at $\nu = 4/150$, as proven in Theorem 8.4. The period is $\tau \approx 0.534173514309816$ and it has been proven that this orbit is unstable. **(Right)** In black, the three Fourier coefficients $(\beta_7(t), \beta_8(t), \beta_9(t))$ of the τ-periodic orbit. In cyan, the rigorously computed two-dimensional unstable normal bundle of the periodic orbit. The apparent rapid change in the bundle is only an artifact of the projection. Indeed, the bundle is smooth over the period of the periodic orbit.

Theorem 8.5. *Let* $\nu = 0.0266$, $\boldsymbol{m} = (m_1, m_2) = (45, 80)$, $\boldsymbol{M} = (M_1, M_2) = (95, 80)$ *and* $\boldsymbol{s} = (s_1, s_2) = (1.4, 1.2)$. *Consider* $\bar{x} = (\bar{L}, \bar{a}, \bar{b}) \in \mathbb{R}^{7032}$ *given in the file* `x0pt0266.txt` *and consider the approximation* $\bar{u}(t, y)$ *given in* (132). *Then there exists a function* $\tilde{u}(t, y)$, *a classical solution of* (1) *such that* $\tilde{u}$ *is periodic with respect to* t *with period* $\tau \in 0.321226891718833 + \delta[-1, 1]$ *with* $\delta = 8.152573 \times 10^{-15}$ *and*

$$\|\tilde{u}(0, \cdot) - \bar{u}(0, \cdot)\|_{C^0} \leq 8.72278 \times 10^{-11}, \qquad \text{and} \qquad \|\tilde{u}(0, \cdot) - \bar{u}(0, \cdot)\|_{L^2} \leq 1.38953 \times 10^{-11}.$$

Let $p = 2\tau$. *Then there exists an eigenvalue-eigenvector pair* (λ, v) *of* (90) *with* $v(t + p, y) = v(t, y)$ *for all* $t \in \mathbb{R}$ *and for all* $y \in [0, 2\pi]$ *such that* $\lambda > 0$ *with* $\lambda \approx 5.031359038130527$. *Since* $|e^{\lambda p}| > 1$, *then the periodic orbit is unstable. The periodic orbit as well as a visualization of the unstable bundle associated to the eigenvector* v *are depicted in Figure 5.*

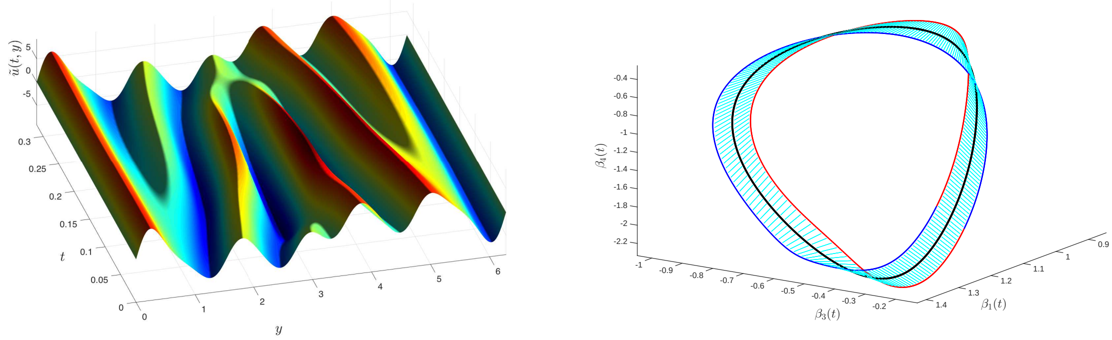

Figure 5: **(Left)** The unstable periodic orbit of (1) at $\nu = 0.0266$ as proved in Theorem 8.5. **(Right)** The three Fourier coefficients $(\beta_1(t), \beta_3(t), \beta_4(t))$ of the τ-periodic orbit of (1) at $\nu = 0.0266$ with $\tau \approx 0.321226891718828$. In cyan, the unstable normal bundle of the periodic orbit as computed in Theorem 8.5. Each "side" of the bundle has a different colour (i.e. blue and red). One can see from this finite dimensional projection that the eigenvector is 2τ-periodic and not τ-periodic.

Theorem 8.6. *Let $\nu = 0.024$, $\boldsymbol{m} = (m_1, m_2) = (80, 85)$, $\boldsymbol{M} = (M_1, M_2) = (190, 92)$ and $\boldsymbol{s} = (s_1, s_2) = (1.4, 1.2)$. Consider $\bar{x} = (\bar{L}, \bar{a}, \bar{b}) \in \mathbb{R}^{13357}$ given in the file* `x0pt024.txt` *and consider the approximation $\bar{u}(t, y)$ given in (132). Then there exists a function $\tilde{u}(t, y)$, a classical solution of (1) such that $\tilde{u}$ is periodic with respect to t with period $\tau \in 0.590231732991113 + \delta[-1, 1]$ with $\delta = 9.542892 \times 10^{-14}$ and*

$$\|\tilde{u}(0, \cdot) - \bar{u}(0, \cdot)\|_{C^0} \leq 3.02426 \times 10^{-10}, \qquad \text{and} \qquad \|\tilde{u}(0, \cdot) - \bar{u}(0, \cdot)\|_{L^2} \leq 4.81759 \times 10^{-11}.$$

Moreover, there exists a positive real Floquet exponent (in fact a Lyapunov exponent) λ with $\lambda \approx 1.429704455017047$ associated to the periodic orbit. Since $|e^{\lambda\tau}| > 1$, then the periodic orbit is unstable. A visualization of the unstable bundle is depicted in Figure 6.

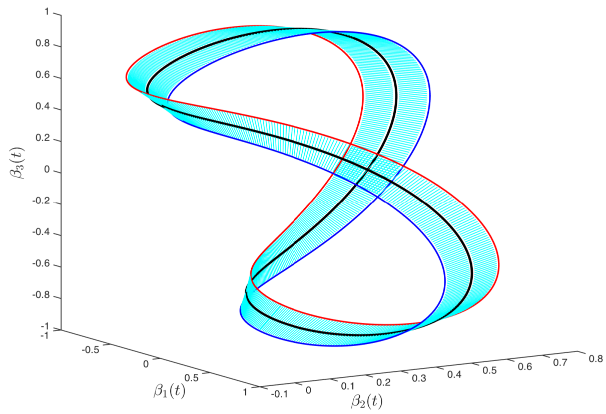

Figure 6: In black, the three Fourier coefficients $(\beta_1(t), \beta_2(t), \beta_3(t))$ of the τ-periodic orbit of (1) at $\nu = 0.024$ with $\tau \approx 0.578517950173901$. In cyan, the unstable normal bundle of the periodic orbit as computed in Theorem 8.6. Each "side" of the bundle has a different colour (i.e. blue and red), and one can see that the eigenvector is τ-periodic as the orbit.

Theorem 8.7. *Let $\nu = 0.0225$, $\boldsymbol{m} = (m_1, m_2) = (70, 85)$, $\boldsymbol{M} = (M_1, M_2) = (155, 88)$ and $\boldsymbol{s} = (s_1, s_2) = (1.4, 1.2)$. Consider $\bar{x} = (\bar{L}, \bar{a}, \bar{b}) \in \mathbb{R}^{n(\boldsymbol{m})} = \mathbb{R}^{11677}$ given in the file* `x0pt0225.txt`*, and let $\bar{u}(t, y)$ the associate approximation given by (132). Then there exists a function $\tilde{u}(t, y)$, a classical solution of (1) such that $\tilde{u}$ is periodic with respect to t with period $\tau \in 0.541896528414205 + \delta[-1, 1]$, with $\delta = 1.54559 \times 10^{-14}$, and such that*

$$\|\tilde{u}(0, \cdot) - \bar{u}(0, \cdot)\|_{C^0} \leq 5.81092 \times 10^{-11}, \qquad \text{and} \qquad \|\tilde{u}(0, \cdot) - \bar{u}(0, \cdot)\|_{L^2} \leq 9.25668 \times 10^{-12}.$$

The periodic orbit is portrayed in Figure 7.

The periodic orbit of Theorem 8.7 is apparently stable, but we do not have a proof of this statement.

Theorem 8.8. *Let $\nu = 0.111485$, $\boldsymbol{m} = (m_1, m_2) = (135, 33)$, $\boldsymbol{M} = (M_1, M_2) = (610, 34)$ and $\boldsymbol{s} = (s_1, s_2) = (1.4, 1.2)$. Consider $\bar{x} = (\bar{L}, \bar{a}, \bar{b}) \in \mathbb{R}^{n(\boldsymbol{m})} = \mathbb{R}^{9147}$ given in the file* `x_0pt111485_gamma_hopf.txt`*, and let $\bar{u}(t, y)$ the associate approximation given by (132). Then there exists a function $\tilde{u}(t, y)$, a classical solution of (1) such that $\tilde{u}$ is periodic with respect to t with period $\tau \in 20.063697521371836 + \delta[-1, 1]$, with $\delta = 2.51984 \times 10^{-6}$, and such that*

$$\|\tilde{u}(0, \cdot) - \bar{u}(0, \cdot)\|_{C^0} \leq 9.21181 \times 10^{-6}, \qquad \text{and} \qquad \|\tilde{u}(0, \cdot) - \bar{u}(0, \cdot)\|_{L^2} \leq 1.46743 \times 10^{-6}.$$

The periodic orbit is portrayed in Figure 8.

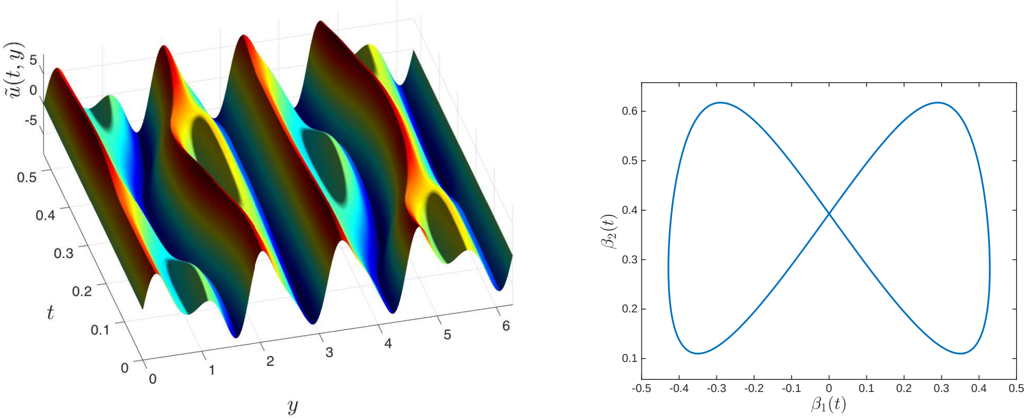

Figure 7: **(Left)** The periodic orbit of (1) at $\nu = 0.0225$ as proven in Theorem 8.7. **(Right)** The Fourier modes $(\beta_1(t), \beta_2(t))$, where the $\beta_{k_2}(t)$ are given by (139).

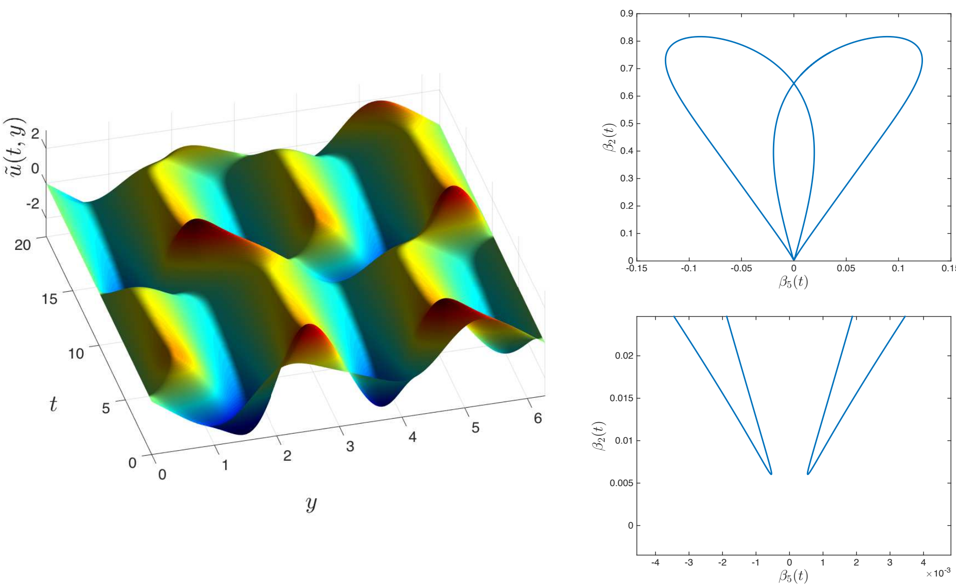

Figure 8: **(Left)** A periodic orbit at $\nu = 0.111485$ of Theorem 8.8. The time period τ is greater than 20. **(Top right)** The two Fourier coefficients $(\beta_2(t), \beta_5(t))$ of the τ-periodic orbit, where the $\beta_{k_2}(t)$ are given by (139). **(Bottom right)** A zoom-in of the same orbit.

A Estimates

A.1 1d estimates

Recall that for a vector $a = (a_k)_{k\in\mathbb{Z}}$, $\|a\|_s = \sup_{k\in\mathbb{Z}} |a_k|\omega_k^s$, where the 1d weights ω_k^s are given by (19).

Lemma A.1 (1d estimates). *Let $M \geq 10$ and $K \geq 1$ computational parameters, and let $s \in (1, 1.45]$.*

Define

$$\alpha_{k,s} \stackrel{\text{def}}{=} \begin{cases} 1 + 2\sum_{j=1}^{K} \frac{1}{j^{2s}} + \frac{2}{(2s-1)K^{2s-1}}, & \text{if } k = 0 \\ 2 + 2\sum_{j=1}^{K} \frac{k^s}{j^s(k+j)^s} + \frac{2k^s}{(k+K+1)^s(s-1)K^{s-1}} + \sum_{j=1}^{k-1} \frac{k^s}{j^s\,(k-j)^s}, & \text{if } 1 \le k < M \\ 2 + 4\sum_{j=1}^{K} \frac{1}{j^s} + \frac{4}{(s-1)K^{s-1}}, & \text{if } k \ge M \end{cases} \tag{134}$$

and

$$\delta_k^{(i,j)} \stackrel{\text{def}}{=} \begin{cases} 1 - \delta_{1,i}\delta_{1,j}, & \text{if } k = 0 \\ 2 - \delta_{1,i} - \delta_{1,j}, & \text{if } k \ge 1. \end{cases}$$

Define

$$\alpha_{k,s}^{(i,j)} = \alpha_{k,s} - \delta_k^{(i,j)}. \tag{135}$$

Then, for all $k \in \mathbb{Z}$

$$\sum_{k_1+k_2=k} \frac{1}{\omega_{k_1}^s \omega_{k_2}^s} \le \frac{\alpha_{k,s}^{(1,1)}}{\omega_k^s}$$

$$\sum_{\substack{k_1+k_2=k \\ k_1 \neq 0}} \frac{1}{\omega_{k_1}^s \omega_{k_2}^s} \le \frac{\alpha_{k,s}^{(0,1)}}{\omega_k^s}$$

$$\sum_{\substack{k_1+k_2=k \\ k_1,k_2 \neq 0}} \frac{1}{\omega_{k_1}^s \omega_{k_2}^s} \le \frac{\alpha_{k,s}^{(0,0)}}{\omega_k^s}.$$

The proof is a slight modification of the estimates in Proposition 1 in [20]. The hypothesis that $M \ge 40$ is taken large enough so that the estimates in [20] hold. See Remark 1 in [20] for more details.

Since for the cases $k \ge M \ge 10$, the bound (135) does not depend on k, denote, given a *large* computational parameter $K > 1$,

$$\alpha_{\infty,s}^{(i,j)} \stackrel{\text{def}}{=} \delta_{1,i} + \delta_{1,j} + 4\sum_{j=1}^{K} \frac{1}{j^s} + \frac{4}{(s-1)K^{s-1}}. \tag{136}$$

A.2 2d estimates

Lemma A.2 (2d estimates). *Let* $a = (a_{\boldsymbol{k}})_{\boldsymbol{k} \in \mathbb{Z}^2}$ *such that* $a_{k_1,0} = a_{0,k_2} = 0$ *for all* $k_1, k_2 \in \mathbb{Z}$*, and* $b = (b_{\boldsymbol{k}})_{\boldsymbol{k} \in \mathbb{Z}^2}$ *such that* $b_{k_1,0} = 0$ *for all* $k_1 \in \mathbb{Z}$*. Recall* (23)*, and assume that* $a, b \in \ell_{\boldsymbol{s}}^\infty$*, that is*

$$\|a\|_{\boldsymbol{s}} = \sup_{\boldsymbol{k} \in \mathbb{Z}^2} \omega_{\boldsymbol{k}}^{\boldsymbol{s}} |a_{\boldsymbol{k}}| < \infty \quad \text{and} \quad \|b\|_{\boldsymbol{s}} = \sup_{\boldsymbol{k} \in \mathbb{Z}^2} \omega_{\boldsymbol{k}}^{\boldsymbol{s}} |b_{\boldsymbol{k}}| < \infty.$$

$$|(a*a)_{\boldsymbol{k}}| \le \frac{\alpha_{k_1,s_1}^{(0,0)} \alpha_{k_2,s_2}^{(0,0)}}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \|a\|_{\boldsymbol{s}}^2, \quad |(a*b)_{\boldsymbol{k}}| \le \frac{\alpha_{k_1,s_1}^{(0,1)} \alpha_{k_2,s_2}^{(0,0)}}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \|a\|_{\boldsymbol{s}} \|b\|_{\boldsymbol{s}}, \quad |(b*b)_{\boldsymbol{k}}| \le \frac{\alpha_{k_1,s_1}^{(1,1)} \alpha_{k_2,s_2}^{(0,0)}}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \|b\|_{\boldsymbol{s}}^2. \tag{137}$$

Proof. Define

$$\mathcal{S}_a = \{\boldsymbol{k} = (k_1, k_2) \ : \ k_1 \neq 0 \text{ and } k_2 \neq 0\} \quad \text{and} \quad \mathcal{S}_b = \{\boldsymbol{k} = (k_1, k_2) \ : \ k_2 \neq 0\}$$

the set of indices on which a and b have respectively non zero entries. Then

$$\begin{aligned}|(a*a)_{\boldsymbol{k}}| &= \left|\sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2=\boldsymbol{k}\\ \boldsymbol{k}^1,\boldsymbol{k}^2\in\mathbb{Z}^2}} a_{\boldsymbol{k}^1}a_{\boldsymbol{k}^2}\right| \le \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2=\boldsymbol{k}\\ \boldsymbol{k}^1\in\mathcal{S}_a,\boldsymbol{k}^2\in\mathcal{S}_a}} \frac{1}{\omega^{\boldsymbol{s}}_{\boldsymbol{k}^1}\omega^{\boldsymbol{s}}_{\boldsymbol{k}^2}}\|a\|^2_{\boldsymbol{s}} = \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2=\boldsymbol{k}\\ \boldsymbol{k}^1\in\mathcal{S}_a,\boldsymbol{k}^2\in\mathcal{S}_a}} \frac{1}{\omega^{s_1}_{k^1_1}\omega^{s_1}_{k^2_1}}\cdot\frac{1}{\omega^{s_2}_{k^1_2}\omega^{s_2}_{k^2_2}}\|a\|^2_{\boldsymbol{s}}\\ &= \left(\sum_{\substack{k^1_1+k^2_1=k_1\\ k^1_1,k^2_1\neq 0}} \frac{1}{\omega^{s_1}_{k^1_1}\omega^{s_1}_{k^2_1}}\right)\left(\sum_{\substack{k^1_2+k^2_2=k_2\\ k^1_2,k^2_2\neq 0}} \frac{1}{\omega^{s_2}_{k^1_2}\omega^{s_2}_{k^2_2}}\right)\|a\|^2_{\boldsymbol{s}} \le \frac{\alpha^{(0,0)}_{k_1,s_1}\alpha^{(0,0)}_{k_2,s_2}}{\omega^{\boldsymbol{s}}_{\boldsymbol{k}}}\|a\|^2_{\boldsymbol{s}},\end{aligned}$$

where the last inequality follows from Lemma A.1. Similarly,

$$\begin{aligned}|(a*b)_{\boldsymbol{k}}| &= \left|\sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2=\boldsymbol{k}\\ \boldsymbol{k}^1,\boldsymbol{k}^2\in\mathbb{Z}^2}} a_{\boldsymbol{k}^1}b_{\boldsymbol{k}^2}\right| \le \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2=\boldsymbol{k}\\ \boldsymbol{k}^1\in\mathcal{S}_a,\boldsymbol{k}^2\in\mathcal{S}_b}} \frac{1}{\omega^{\boldsymbol{s}}_{\boldsymbol{k}^1}\omega^{\boldsymbol{s}}_{\boldsymbol{k}^2}}\|a\|_{\boldsymbol{s}}\|b\|_{\boldsymbol{s}} = \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2=\boldsymbol{k}\\ \boldsymbol{k}^1\in\mathcal{S}_a,\boldsymbol{k}^2\in\mathcal{S}_b}} \frac{1}{\omega^{s_1}_{k^1_1}\omega^{s_1}_{k^2_1}}\cdot\frac{1}{\omega^{s_2}_{k^1_2}\omega^{s_2}_{k^2_2}}\|a\|_{\boldsymbol{s}}\|b\|_{\boldsymbol{s}}\\ &= \left(\sum_{\substack{k^1_1+k^2_1=k_1\\ k^1_1\neq 0}} \frac{1}{\omega^{s_1}_{k^1_1}\omega^{s_1}_{k^2_1}}\right)\left(\sum_{\substack{k^1_2+k^2_2=k_2\\ k^1_2,k^2_2\neq 0}} \frac{1}{\omega^{s_2}_{k^1_2}\omega^{s_2}_{k^2_2}}\right)\|a\|_{\boldsymbol{s}}\|b\|_{\boldsymbol{s}} \le \frac{\alpha^{(0,1)}_{k_1,s_1}\alpha^{(0,0)}_{k_2,s_2}}{\omega^{\boldsymbol{s}}_{\boldsymbol{k}}}\|a\|_{\boldsymbol{s}}\|b\|_{\boldsymbol{s}},\end{aligned}$$

and

$$\begin{aligned}|(b*b)_{\boldsymbol{k}}| &= \left|\sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2=\boldsymbol{k}\\ \boldsymbol{k}^1,\boldsymbol{k}^2\in\mathbb{Z}^2}} b_{\boldsymbol{k}^1}b_{\boldsymbol{k}^2}\right| \le \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2=\boldsymbol{k}\\ \boldsymbol{k}^1\in\mathcal{S}_b,\boldsymbol{k}^2\in\mathcal{S}_b}} \frac{1}{\omega^{\boldsymbol{s}}_{\boldsymbol{k}^1}\omega^{\boldsymbol{s}}_{\boldsymbol{k}^2}}\|b\|^2_{\boldsymbol{s}} = \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2=\boldsymbol{k}\\ \boldsymbol{k}^1\in\mathcal{S}_b,\boldsymbol{k}^2\in\mathcal{S}_b}} \frac{1}{\omega^{s_1}_{k^1_1}\omega^{s_1}_{k^2_1}}\cdot\frac{1}{\omega^{s_2}_{k^1_2}\omega^{s_2}_{k^2_2}}\|b\|^2_{\boldsymbol{s}}\\ &= \left(\sum_{k^1_1+k^2_1=k_1} \frac{1}{\omega^{s_1}_{k^1_1}\omega^{s_1}_{k^2_1}}\right)\left(\sum_{\substack{k^1_2+k^2_2=k_2\\ k^1_2,k^2_2\neq 0}} \frac{1}{\omega^{s_2}_{k^1_2}\omega^{s_2}_{k^2_2}}\right)\|b\|^2_{\boldsymbol{s}} \le \frac{\alpha^{(1,1)}_{k_1,s_1}\alpha^{(0,0)}_{k_2,s_2}}{\omega^{\boldsymbol{s}}_{\boldsymbol{k}}}\|b\|^2_{\boldsymbol{s}}. \qquad \square\end{aligned}$$

Remark A.3. For $s \in (1, 1.45)$, $\alpha_{k,s}$ as defined in Lemma A.1 satisfy $\alpha_{k,s} \le \alpha_{k+1,s}$ for all $k \ge 0$.

As a consequence of the previous remark, we have the following asymptotic estimate.

Lemma A.4 (Asymptotic Estimates). *Assume that all hypotheses of Lemma A.2 hold. Denote* $\boldsymbol{M} = (M_1, M_2) \succeq (40, 40)$. *Given* $\boldsymbol{k} \notin \boldsymbol{F_M}$, *we have that*

$$|(a*a)_{\boldsymbol{k}}| \le \frac{\alpha^{(0,0)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2}}{\omega^{\boldsymbol{s}}_{\boldsymbol{k}}}\|a\|^2_{\boldsymbol{s}}, \quad |(a*b)_{\boldsymbol{k}}| \le \frac{\alpha^{(0,1)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2}}{\omega^{\boldsymbol{s}}_{\boldsymbol{k}}}\|a\|_{\boldsymbol{s}}\|b\|_{\boldsymbol{s}}, \quad |(b*b)_{\boldsymbol{k}}| \le \frac{\alpha^{(1,1)}_{M_1,s_1}\alpha^{(0,0)}_{M_2,s_2}}{\omega^{\boldsymbol{s}}_{\boldsymbol{k}}}\|b\|^2_{\boldsymbol{s}}. \tag{138}$$

B Visualizing the orbit with the time Fourier coefficients

It is common to visualize the solutions in a state space of Fourier coefficients. To do this, rewrite the Fourier expansion (2) as

$$u(t,y) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} c_{\boldsymbol{k}} e^{iLk_1t}e^{ik_2y} = \sum_{k_2\in\mathbb{Z}} \overbrace{\left(\sum_{k_1\in\mathbb{Z}} c_{k_1,k_2}e^{iLk_1t}\right)}^{\stackrel{\text{def}}{=}\alpha_{k_2}(t)} e^{ik_2y}.$$

Recalling the third relation of (3), i.e. $c_{-k_1,k_2} = -\text{conj}(c_{\boldsymbol{k}})$ and the third relation of (4) i.e. $a_{0,k_2} = Re(c_{0,k_2}) = 0$ for all $k_2 \geq 0$,

$$\begin{aligned}\alpha_{k_2}(t) &= \sum_{k_1 \in \mathbb{Z}} c_{k_1,k_2} e^{iLk_1 t} \\ &= c_{0,k_2} - \sum_{k_1=1}^{\infty} \text{conj}(c_{k_1,k_2}) e^{-iLk_1 t} + \sum_{k_1=1}^{\infty} c_{k_1,k_2} e^{iLk_1 t} \\ &= i \left[b_{0,k_2} + 2 \sum_{k_1=1}^{\infty} \left(a_{k_1,k_2} \sin(k_1 Lt) + b_{k_1,k_2} \cos(k_1 Lt) \right) \right].\end{aligned}$$

Letting

$$\beta_{k_2}(t) \stackrel{\text{def}}{=} b_{0,k_2} + 2 \sum_{k_1=1}^{\infty} \left(a_{k_1,k_2} \sin(k_1 Lt) + b_{k_1,k_2} \cos(k_1 Lt) \right) \tag{139}$$

we get using (3) that $\beta_{-k_2} = -\beta_{k_2}$ and therefore that

$$u(t, y) = -2 \sum_{k_2=1}^{\infty} \beta_{k_2}(t) \sin(k_2 y).$$

Acknowledgments

MG was partially supported by FAPESP grants 2013/07460-7, 2013/50382-7 and 2010/00875-9, and by CNPq grants 305860/2013-5 and 306453/2009-6, Brazil.

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