Analytic Estimates and Rigorous Continuation for Equilibria of Higher-Dimensional PDEs

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Abstract

In this paper we extend the ideas of the so-called *validated continuation* technique to the context of rigorously proving the existence of equilibria for partial differential equations defined on higher-dimensional spatial domains. For that effect we present a new set of general analytic estimates. These estimates are valid for any dimension and are used, together with rigorous computations, to construct a finite number of *radii polynomials*. These polynomials provide a computationally efficient method to prove, via a contraction argument, the existence and local uniqueness of solutions for a rather large class of nonlinear problems. We apply this technique to prove existence and local uniqueness of equilibrium solutions for the Cahn-Hilliard and the Swift-Hohenberg equations defined on two- and three-dimensional spatial domains.

1 Introduction

Partial differential equations (PDEs) arising in fluid dynamics and material science are naturally defined on two- and three-dimensional spatial domains. With the extensive use of PDE modeling in engineering, developing new mathematical tools to study rigorously these equations is of central importance in science. However, analytically finding solutions of nonlinear PDEs is generally an extremely difficult problem. The availability of powerful computers and sophisticated software then makes numerical simulation the primary tool for scientists and engineers confronted with nonlinear problems. In particular, one of the most efficient methods for determining equilibria of a parameter dependent PDE

$$u_t = E(u, \lambda), \quad \lambda \in \mathbb{R} \tag{1}$$

is to use a predictor-corrector continuation algorithm. Since (1) is infinite dimensional, the numerical method is applied to a finite dimensional approximation. This raises the natural question of the validity of the output. How does one make sure that the truncation error term induced by computing on a finite projection did not lead to spurious branches of solutions? In order to address this question, several computer-assisted proofs of existence of solutions of nonlinear PDEs were presented in the last decade (see for example [1, 2, 3, 4, 5, 6]). These proofs are based on local topological arguments like the non vanishing of the Conley Index of a small isolating neighborhood of the solution or on a contraction mapping argument, both of which rely on the fact that the linear part of the PDE governs, at least locally, the behavior of the system. The most fundamental problem in developing these rigorous numerical methods is to control the truncation error terms by building sharp enough

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analytic estimates so that the computer can be used to verify that the nonlinear part is locally negligible. In the above cited papers, the bounds on the truncation errors are given in terms of regularity conditions on the solutions. In [2, 3, 7, 6] and in [4, 5], such estimates are presented for PDEs defined on one- and two-dimensional rectangular spatial domains, respectively. In this paper, we present new analytic estimates for PDEs defined on $d-$dimensional spatial domains. To the best of our knowledge, this is the first attempt to present general estimates in this context. We then use these general estimates to extend the ideas of the so-called *validated continuation* method to the context of rigorously proving the existence of equilibria for PDEs defined on higher-dimensional domains. It is important to mention that validated continuation was originally introduced as a semi-rigorous numerical method, where some of the computations are allowed to be non-rigorous. In this way, the results of the computations are not computer-assisted proofs due to round-off errors (see [3]). In the present work, we extend the ideas of validated continuation to PDEs in higher dimensions, and we make all computations rigorous by using interval arithmetic. In this way we produce completely rigorous computer-assisted proofs. Hence, we refer to this method as *rigorous continuation.* Although we present the theory in the context of finding equilibria of PDEs, the method should be applicable to rigorously compute equilibria for systems of PDEs and time periodic solutions of PDEs.

Validated continuation was introduced in [3] and later on improved in [8] to compute discrete branches of equilibria of (1), when the PDE is defined on a one dimensional (interval) spatial domain. Combining the information obtained from the predictor-corrector steps with rigorous interval arithmetic computations and analytic estimates, this rigorous numerical method verifies that the numerically produced equilibrium solution for the finite dimensional system can be used to explicitly define a set which contains a unique solution for the infinite dimensional problem. In [9, 10, 11], validated continuation was adapted to compute global smooth branches of time periodic solutions of delay differential equations and ordinary differential equations. As mentioned earlier, the main focus of the present work is to develop a set of consistent general analytic estimates in order to prove existence of steady state solutions of (1). To make things more precise, we assume that $E(\cdot,\lambda)$ is a densely defined operator on a Hilbert space H, and is explicitly given by

$$u_t = L(u,\lambda) + \sum_{n=2}^{p} q_n u^n \tag{2}$$

in a domain $\Omega \subset \mathbb{R}^d$, where $\lambda \in \mathbb{R}$ is a parameter, $L = L(\cdot,\lambda)\colon D(L) \subset H \to H$ is a parameter dependent linear operator, and $q_n = q_n(\lambda) \in \mathbb{R}$ are the coefficients of the polynomial nonlinearity of degree p. We also assume that H has an orthogonal basis $\{\psi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ formed by eigenfunctions of L, which are assumed to be independent of λ, and that the domain of L is given by

$$D(L) = \left\{ u = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c_{\boldsymbol{k}}\psi_{\boldsymbol{k}} \in H \;\middle|\; \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \mu_{\boldsymbol{k}} c_{\boldsymbol{k}} \psi_{\boldsymbol{k}} \;\text{ converges} \right\},$$

where $\mu_{\boldsymbol{k}} = \mu_{\boldsymbol{k}}(\lambda)$ are the eigenvalues of $L(\cdot,\lambda)$. Then the expansion of (2) in terms of the basis $\{\psi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ takes the form

$$\dot{c}_{\boldsymbol{k}} = \mu_{\boldsymbol{k}} c_{\boldsymbol{k}} + \sum_{n=2}^{p} q_n \sum_{\substack{\boldsymbol{k}^1+\cdots+\boldsymbol{k}^n=\boldsymbol{k} \\ \boldsymbol{k}^j\in\mathbb{Z}^d}} c_{\boldsymbol{k}^1}\cdots c_{\boldsymbol{k}^n}, \tag{3}$$

with $\boldsymbol{k} = (k_1,\dots,k_d) \in \mathbb{Z}^d$, where $\boldsymbol{k}^j = (k_1^j,\dots,k_d^j) \in \mathbb{Z}^d$ for $1 \le j \le n$, and $\dot{c}_{\boldsymbol{k}} = \frac{d}{dt} c_{\boldsymbol{k}}$. Defining the vector of a priori unknown coefficients by $c := \{c_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$, when looking for equilibrium solutions of (3), we need to solve

$$f_{\boldsymbol{k}}(c,\lambda) := \mu_{\boldsymbol{k}} c_{\boldsymbol{k}} + \sum_{n=2}^{p} q_n \sum_{\substack{\boldsymbol{k}^1+\cdots+\boldsymbol{k}^n=\boldsymbol{k} \\ \boldsymbol{k}^j\in\mathbb{Z}^d}} c_{\boldsymbol{k}^1}\cdots c_{\boldsymbol{k}^n} = 0, \tag{4}$$

for every $\boldsymbol{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d$. Denoting $f := \{f_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{Z}^d}$ we show, in Section 3, that solving the infinite dimensional problem

$$f(x, \lambda) = 0 \tag{5}$$

for $x := \{x_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{Z}^d}$ in a Banach space $X^{\boldsymbol{s}}$ of fast decaying coefficients is equivalent to looking for equilibrium solutions of (2). The theoretical justification for this choice of Banach space lies is the fact that the solutions we are looking for have sufficient regularity.

Remark 1.1. *All the results presented in this paper are valid if the coefficients* $\{c_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{Z}^d}$ *in the expansion (3) are complex. This is the case if the PDE (2) is complex or if we use complex-valued functions as basis elements, like complex exponentials as Fourier basis for example. In that case we can split* $c_{\boldsymbol{k}}$ *in its real and imaginary parts and define (4) for*

$$\tilde{c}_{\boldsymbol{k}} := \begin{pmatrix} \mathrm{Re}(c_{\boldsymbol{k}}) \\ \mathrm{Im}(c_{\boldsymbol{k}}) \end{pmatrix},$$

and define

$$\tilde{f}_{\boldsymbol{k}} := \begin{pmatrix} \mathrm{Re}(f_{\boldsymbol{k}}) \\ \mathrm{Im}(f_{\boldsymbol{k}}) \end{pmatrix}.$$

In such case, the formulas presented in Section 3 have two components and the inequalities should be understood as component-wise.

Next we describe in detail the space $X^{\boldsymbol{s}}$, but first we need to introduce some notation. As already suggested above, we use boldface type to denote multi-indices as in $\boldsymbol{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d$. We denote by $|\cdot|$ the component-wise absolute value, that is, $|\boldsymbol{k}| := (|k_1|, \dots, |k_d|)$. Given $\boldsymbol{k}, \boldsymbol{n} \in \mathbb{Z}^d$ we also use component-wise inequalities. So that $\boldsymbol{k} < \boldsymbol{n}$, for example, means that $k_j < n_j$ for all $1 \le j \le d$. Similarly for $\boldsymbol{k} \le \boldsymbol{n}$, $\boldsymbol{k} > \boldsymbol{n}$, and $\boldsymbol{k} \ge \boldsymbol{n}$. Throughout this paper $\boldsymbol{m} = (m_1, \dots, m_d)$ and $\boldsymbol{M} = (M_1, \dots, M_d)$ always denote computational parameters such that $\boldsymbol{M} \ge \boldsymbol{m}$, and $M_j \ge 6$ for all $1 \le j \le d$. Also $\boldsymbol{s} = (s_1, \dots, s_d)$ always denote the "decay rate", where each s_j is the *decay rate* on the jth-coordinate, and is such that $s_j \ge 2$ for all $1 \le j \le d$. We also denote the *finite set of indices* of "sizes" $\boldsymbol{m}$ and $\boldsymbol{M}$ respectively by $\boldsymbol{F_m} := \{\boldsymbol{k} \in \mathbb{Z}^d \mid |\boldsymbol{k}| < \boldsymbol{m}\}$ and $\boldsymbol{F_M} := \{\boldsymbol{k} \in \mathbb{Z}^d \mid |\boldsymbol{k}| < \boldsymbol{M}\}$. Notice that $\boldsymbol{F_m} = F_{m_1} \times \cdots \times F_{m_d}$, where $F_{m_j} := \{k_j \in \mathbb{Z} \mid |k_j| < m_j\}$. Similarly for $\boldsymbol{F_M}$.

We now describe the space $X^{\boldsymbol{s}}$. Recalling the definition of the one-dimensional weights ω_k^s in (41) from Appendix A, we define the d-dimensional weights

$$\omega_{\boldsymbol{k}}^{\boldsymbol{s}} := \prod_{j=1}^{d} \omega_{k_j}^{s_j},$$

which are used to define the norm

$$\|x\|_{\boldsymbol{s}} = \sup_{\boldsymbol{k} \in \mathbb{Z}^d} \omega_{\boldsymbol{k}}^{\boldsymbol{s}} |x_{\boldsymbol{k}}|,$$

and the Banach space

$$X^{\boldsymbol{s}} = \{x \mid \|x\|_{\boldsymbol{s}} < \infty\}, \tag{6}$$

consisting of sequences with algebraically decaying tails according to the rate $\boldsymbol{s}$. We look for solutions of (5) within balls $B \subset X^{\boldsymbol{s}}$ of radius r (with respect to the norm $\|\cdot\|_{\boldsymbol{s}}$). The idea of rigorous continuation is to construct a parameter dependent contraction $T_\lambda \colon B \to B$ and to use a contraction mapping theorem to conclude the existence of a unique solution of $f(x, \lambda) = 0$ within the set B. The contraction rate of T_λ depends on the magnitude of the eigenvalues of $L(\cdot, \lambda)$. The verification of the contraction depends on a subtle balance between the growth of the eigenvalues and our control on the truncation error, provided by analytic estimates. The slower the eigenvalues grow, the sharper the analytic estimates on the nonlinear truncation error terms need to be. The construction of the estimates is done in Section 2. In order to verify the hypotheses of a contraction argument in a computationally efficient way, we recall the notion of *radii polynomials* [9, 10, 3, 8, 11]. The independent variable of the polynomials is the radius r of the ball B. In essence, we solve for the set B, by finding a radius r that makes all the radii polynomials simultaneously negative. A brief discussion on these polynomials and the theory of rigorous continuation in done in Section 3. In Section 4 we explicitly construct the radii polynomials for the case of a cubic nonlinerarity. Finally, in Section 5 we present applications of the method to the Cahn-Hilliard and the Swift-Hohenbergh PDEs defined on two- and three-dimensional spatial domains. It is worth emphasizing that for each of the computed solutions, we have a computed-assisted proof of existence and local uniqueness of an equilibrium for the PDE.

2 Analytic Estimates

One of the fundamental step for the computation of (steady state) solutions of partial differential equations is to build sharp enough analytic estimates on the nonlinear terms. In particular, the nonlinear part of (5) involves convolution terms of the form

$$\left(c^{(1)} * \cdots * c^{(n)}\right)_{\boldsymbol{k}} := \sum_{\substack{\boldsymbol{k}^1+\cdots+\boldsymbol{k}^n=\boldsymbol{k} \\ \boldsymbol{k}^j \in \mathbb{Z}^d}} c^{(1)}_{\boldsymbol{k}^1} \cdots c^{(n)}_{\boldsymbol{k}^n}, \tag{7}$$

where each $c^{(j)} = \{c^{(j)}_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is a sequence of real (or complex) numbers indexed by $\boldsymbol{k} \in \mathbb{Z}^d$. As mentioned earlier, the bounds on the truncation errors are given in terms of regularity conditions on the solutions. More precisely, assuming that each coefficient $c^{(j)}_{\boldsymbol{k}}$ goes to zero with a certain decay rate, one shows that the convolution term (7) goes to zero with the same decay. These general asymptotic bounds are presented in Section 2.1. In Section 2.2, we consider the computational parameter $\boldsymbol{M}$ which provides a computational alternative to improve the general bounds of Section 2.1. More specifically, for $\boldsymbol{k} \in \boldsymbol{F_M}$ one splits (7) into a finite sum of *"size"* $\boldsymbol{M}$ that we explicitly compute using interval arithmetic and an infinite sum that we bound using analytic estimates. Finally, in Section 2.3, we consider the case $\boldsymbol{k} \notin \boldsymbol{F_M}$ and derive a *uniform* asymptotic bound for (7). Using this uniform bound, the verification of the above mentioned contraction mapping theorem reduces to a *finite* number of computations as is described in Section 3.

2.1 General Estimates

The goal of this section is to generalize the different one-dimensional estimates presented in [9, 2, 3, 8, 6] to the d-dimensional case. These new d-dimensional estimates are constructive and are based on the rather sharp one-dimensional general estimates defined in [9]. These one-dimensional estimates are presented in Appendix A. We present them because we introduce some modifications (see Remark A.5) and also for the sake of completeness. Recalling the definition of $\alpha^{(n)}_k$ from Appendix A we define

$$\alpha^{(n)}_{\boldsymbol{k}} = \alpha^{(n)}_{\boldsymbol{k}}(\boldsymbol{s}, \boldsymbol{M}) := \prod_{j=1}^{d} \alpha^{(n)}_{k_j}(s_j, M_j). \tag{8}$$

The following bounds are given in terms of regularity conditions on the solutions.

Lemma 2.1 (General estimates)**.** *Suppose there exist* $A_1, A_2, \ldots, A_n$ *such that for every* $j \in \{1, \ldots, n\}$ *and every* $\boldsymbol{k} \in \mathbb{Z}^d$*, we have that*

$$\left|c^{(j)}_{\boldsymbol{k}}\right| \le \frac{A_j}{\omega^{\boldsymbol{s}}_{\boldsymbol{k}}}, \tag{9}$$

Then, for any $\boldsymbol{k} \in \mathbb{Z}^d$, we get that

$$\left| \left(c^{(1)} * \cdots * c^{(n)} \right)_{\boldsymbol{k}} \right| \leq \left(\prod_{j=1}^{n} A_j \right) \frac{\alpha_{\boldsymbol{k}}^{(n)}}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}. \tag{10}$$

Proof. We have that

$$\begin{aligned}
\left| \left(c^{(1)} * \cdots * c^{(n)} \right)_{\boldsymbol{k}} \right| &= \left| \sum_{\substack{\boldsymbol{k}^1 + \cdots + \boldsymbol{k}^n = \boldsymbol{k} \\ \boldsymbol{k}^1, \dots, \boldsymbol{k}^n \in \mathbb{Z}^d}} c_{\boldsymbol{k}^1}^{(1)} \cdots c_{\boldsymbol{k}^n}^{(n)} \right| \leq \sum_{\substack{\boldsymbol{k}^1 + \cdots + \boldsymbol{k}^n = \boldsymbol{k} \\ \boldsymbol{k}^1, \dots, \boldsymbol{k}^n \in \mathbb{Z}^d}} \frac{A_1}{\omega_{\boldsymbol{k}^1}^{\boldsymbol{s}}} \cdots \frac{A_n}{\omega_{\boldsymbol{k}^n}^{\boldsymbol{s}}} \\
&= \left(\prod_{j=1}^{n} A_j \right) \left(\sum_{\substack{\boldsymbol{k}^1 + \cdots + \boldsymbol{k}^n = \boldsymbol{k} \\ \boldsymbol{k}^1, \dots, \boldsymbol{k}^n \in \mathbb{Z}^d}} \frac{1}{\omega_{\boldsymbol{k}^1}^{\boldsymbol{s}} \cdots \omega_{\boldsymbol{k}^n}^{\boldsymbol{s}}} \right) \\
&= \left(\prod_{j=1}^{n} A_j \right) \left(\sum_{\substack{\boldsymbol{k}^1 + \cdots + \boldsymbol{k}^n = \boldsymbol{k} \\ \boldsymbol{k}^1, \dots, \boldsymbol{k}^n \in \mathbb{Z}^d}} \prod_{j=1}^{d} \frac{1}{\omega_{k_j^1}^{s_j} \cdots \omega_{k_j^n}^{s_j}} \right) \\
&= \left(\prod_{j=1}^{n} A_j \right) \left(\prod_{j=1}^{d} \sum_{\substack{k_j^1 + \cdots + k_j^n = k_j \\ k_j^1, \dots, k_j^n \in \mathbb{Z}}} \frac{1}{\omega_{k_j^1}^{s_j} \cdots \omega_{k_j^n}^{s_j}} \right) \\
&\leq \left(\prod_{j=1}^{n} A_j \right) \prod_{j=1}^{d} \frac{\alpha_{k_j}^{(n)}}{\omega_{k_j}^{s_j}} = \left(\prod_{j=1}^{n} A_j \right) \frac{\alpha_{\boldsymbol{k}}^{(n)}}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}},
\end{aligned}$$

where the last inequality follows from Lemma A.4. ■

2.2 Refinement for the Case $\boldsymbol{k} \in \boldsymbol{F_M}$

We now present a possible refinement for the general bounds introduced in Section 2.1, by allowing one to do explicit computations. Given sequences $c^{(j)} = \{c_{\boldsymbol{k}}^{(j)}\}_{\boldsymbol{k} \in \mathbb{Z}^d}$ we define $c_{\boldsymbol{F_M}}^{(j)}$, the *finite part* of $c^{(j)}$, component-wise by

$$\left(c_{\boldsymbol{F_M}}^{(j)} \right)_{\boldsymbol{k}} = \begin{cases} c_{\boldsymbol{k}}^{(j)}, & \text{if } \boldsymbol{k} \in \boldsymbol{F_M} \\ 0, & \text{if } \boldsymbol{k} \notin \boldsymbol{F_M} \end{cases}$$

and consider the splitting

$$\left(c^{(1)} * \cdots * c^{(n)} \right)_{\boldsymbol{k}} = \left(c_{\boldsymbol{F_M}}^{(1)} * \cdots * c_{\boldsymbol{F_M}}^{(n)} \right)_{\boldsymbol{k}} + \sum_{\substack{\boldsymbol{k}^1 + \cdots + \boldsymbol{k}^n = \boldsymbol{k} \\ \{\boldsymbol{k}^1, \dots, \boldsymbol{k}^n\} \not\subset \boldsymbol{F_M}}} c_{\boldsymbol{k}^1}^{(1)} \cdots c_{\boldsymbol{k}^n}^{(n)}, \tag{11}$$

where the first term is a finite convolution sum and is explicitly computed using the Fast Fourier Transform (FFT) algorithm and interval arithmetic as described in [8]. We use the following results to bound the infinite sum in the splitting above. Recalling the definition of $\varepsilon_k^{(n)}$ in (42) from Appendix A we define

$$\varepsilon_{\boldsymbol{k}}^{(n)} = \varepsilon_{\boldsymbol{k}}^{(n)}(\boldsymbol{s}, \boldsymbol{M}) := \frac{\alpha_{\boldsymbol{k}}^{(n)}}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \max_{j=1,\dots,d} \left\{ \frac{\omega_{k_j}^{s_j}}{\alpha_{k_j}^{(n)}(s_j, M_j)} \varepsilon_{k_j}^{(n)}(s_j, M_j) \right\}. \tag{12}$$

Lemma 2.2. *Given $\boldsymbol{k} \in \boldsymbol{F_M}$ and assuming that the regularity condition (9) is satisfied, we have*

$$\left| \sum_{\substack{\boldsymbol{k}^1+\cdots+\boldsymbol{k}^n=\boldsymbol{k} \\ \{\boldsymbol{k}^1,\ldots,\boldsymbol{k}^\ell\}\not\subset \boldsymbol{F_M}}} c^{(1)}_{\boldsymbol{k}^1}\cdots c^{(n)}_{\boldsymbol{k}^n} \right| \le \ell \left(\prod_{j=1}^n A_j\right) \varepsilon^{(n)}_{\boldsymbol{k}}.$$

Proof. We have that

$$\begin{aligned}
\left| \sum_{\substack{\boldsymbol{k}^1+\cdots+\boldsymbol{k}^n=\boldsymbol{k} \\ \{\boldsymbol{k}^1,\ldots,\boldsymbol{k}^\ell\}\not\subset \boldsymbol{F_M}}} c^{(1)}_{\boldsymbol{k}^1}\cdots c^{(n)}_{\boldsymbol{k}^n} \right| &\le \left(\prod_{j=1}^n A_j\right) \sum_{\substack{\boldsymbol{k}^1+\cdots+\boldsymbol{k}^n=\boldsymbol{k} \\ \{\boldsymbol{k}^1,\ldots,\boldsymbol{k}^\ell\}\not\subset \boldsymbol{F_M}}} \frac{1}{\omega^{\boldsymbol{s}}_{\boldsymbol{k}^1}}\cdots\frac{1}{\omega^{\boldsymbol{s}}_{\boldsymbol{k}^n}} \\
&\le \ell \left(\prod_{j=1}^n A_j\right) \sum_{\substack{\boldsymbol{k}^1+\cdots+\boldsymbol{k}^n=\boldsymbol{k} \\ \boldsymbol{k}^1\notin \boldsymbol{F_M}}} \frac{1}{\omega^{\boldsymbol{s}}_{\boldsymbol{k}^1}}\cdots\frac{1}{\omega^{\boldsymbol{s}}_{\boldsymbol{k}^n}}
\end{aligned}$$

$$\begin{aligned}
&\le \ell \left(\prod_{j=1}^n A_j\right) \max_{j_0=1,\ldots,d} \left\{ \left(\prod_{\substack{j=1\\ j\neq j_0}}^d \sum_{\substack{k_j^1+\cdots+k_j^n=k_j \\ k_j^1,\ldots,k_j^n\in\mathbb{Z}}} \frac{1}{\omega^{s_j}_{k_j^1}\cdots\omega^{s_j}_{k_j^n}} \right) \sum_{\substack{k^1_{j_0}+\cdots+k^n_{j_0}=k_{j_0} \\ k^1_{j_0}\notin F_{M_{j_0}}}} \frac{1}{\omega^{s_{j_0}}_{k^1_{j_0}}\cdots\omega^{s_{j_0}}_{k^n_{j_0}}} \right\} \\
&\le \ell \left(\prod_{j=1}^n A_j\right) \max_{j_0=1,\ldots,d} \left\{ \left(\prod_{\substack{j=1\\ j\neq j_0}}^d \frac{\alpha^{(n)}_{k_j}}{\omega^{s_j}_{k_j}} \right) \varepsilon^{(n)}_{k_{j_0}} \right\} \\
&= \ell \left(\prod_{j=1}^n A_j\right) \frac{\alpha^{(n)}_{\boldsymbol{k}}}{\omega^{\boldsymbol{s}}_{\boldsymbol{k}}} \max_{j=1,\ldots,d} \left\{ \frac{\omega^{s_j}_{k_j}}{\alpha^{(n)}_{k_j}} \varepsilon^{(n)}_{k_j} \right\} = \ell \left(\prod_{j=1}^n A_j\right) \varepsilon^{(n)}_{\boldsymbol{k}},
\end{aligned}$$

where the last two inequalities follow from Lemma A.4 and Corollary A.6, respectively. ■

Corollary 2.3. *Given $\boldsymbol{k} \in \boldsymbol{F_M}$ and assuming that the regularity condition (9) is satisfied, we have*

$$\left| \left(c^{(1)} * \cdots * c^{(n)}\right)_{\boldsymbol{k}} \right| \le \left| \left(c^{(1)}_{\boldsymbol{F_M}} * \cdots * c^{(n)}_{\boldsymbol{F_M}}\right)_{\boldsymbol{k}} \right| + n \left(\prod_{j=1}^n A_j\right) \varepsilon^{(n)}_{\boldsymbol{k}}.$$

Proof. The result immediately follows from the splitting (11) and Lemma 2.2. ■

2.3 Uniform Estimate for the Case $\boldsymbol{k} \notin \boldsymbol{F_M}$

In this section we present a uniform estimate for the case $\boldsymbol{k} \notin \boldsymbol{F_M}$. For $M \in \mathbb{N}$, with $M \ge 6$ and $s \ge 2$ we define

$$\tilde{\alpha}^{(n)}_M = \tilde{\alpha}^{(n)}_M(s,M) := \max\left\{ \alpha^{(n)}_k(s,M) \mid k = 0,\ldots,M \right\},$$

and then

$$\tilde{\alpha}^{(n)}_{\boldsymbol{M}} = \tilde{\alpha}^{(n)}_{\boldsymbol{M}}(\boldsymbol{s},\boldsymbol{M}) := \max_{j_0=1,\ldots,d} \left\{ \alpha^{(n)}_{M_{j_0}}(s_{j_0}, M_{j_0}) \prod_{\substack{j=1\\ j\neq j_0}}^d \tilde{\alpha}^{(n)}_{M_j}(s_j, M_j) \right\}. \tag{13}$$

We then have the following lemma.

Lemma 2.4. *Given $\boldsymbol{k} \notin \boldsymbol{F_M}$ and assuming that the regularity condition (9) is satisfied, we have*

$$\left|\left(c^{(1)} * \cdots * c^{(n)}\right)_{\boldsymbol{k}}\right| \leq \left(\prod_{j=1}^{n} A_j\right) \frac{\tilde{\alpha}_{\boldsymbol{M}}^{(n)}}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}. \tag{14}$$

Proof. Since $\boldsymbol{k} \notin \boldsymbol{F_M}$, there exists $j_0 \in \{1, \ldots, d\}$ such that $|k_{j_0}| \geq M_{j_0}$. From Remark A.1, this implies that $\alpha_{k_{j_0}}^{(n)} \leq \alpha_{M_{j_0}}^{(n)}$, and therefore

$$\alpha_{\boldsymbol{k}}^{(n)} = \alpha_{k_{j_0}}^{(n)} \prod_{\substack{j=1 \\ j \neq j_0}}^{d} \alpha_{k_j}^{(n)} \leq \alpha_{M_{j_0}}^{(n)} \prod_{\substack{j=1 \\ j \neq j_0}}^{d} \tilde{\alpha}_{M_j}^{(n)} \leq \tilde{\alpha}_{\boldsymbol{M}}^{(n)}.$$

The result then follows from the general estimates given by (10). ∎

3 Rigorous Continuation and Radii Polynomials

Using the general analytic estimates for discrete convolution sums introduced in the previous sections, we can generalize the ideas from the validated continuation method [9, 10, 3, 8, 11] to the context of proving rigorously the existence of equilibria of PDEs defined on d-dimensional domains. We refer to this as *rigorous continuation*. The essential ingredient of this method is the construction of the radii polynomials. Their construction combines the analytic estimates introduced in Section 2 with a computational version of the Banach Fixed Point Theorem applied to subsets of $X^{\boldsymbol{s}}$ (see Lemma 3.3). Hence, the fact that $X^{\boldsymbol{s}}$ is a Banach space is crucial. The proof of the following is standard and it is omitted.

Lemma 3.1. $X^{\boldsymbol{s}} = \{x \mid \|x\|_{\boldsymbol{s}} < \infty\}$ *is a Banach space.*

As mentioned in Section 1, we transform the problem of looking for equilibrium solutions of (2) into the equivalent problem (5). The two problems are equivalent under regularity assumptions on the solution of the PDE. The following lemma makes this precise.

Lemma 3.2. *Assume the following regularity condition on the solutions of (2):* If

$$u = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} c_{\boldsymbol{k}} \psi_{\boldsymbol{k}} \tag{15}$$

is a solution of (2) in H, then $c = \{c_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{Z}^d} \in X^{\boldsymbol{s}}$.

Under this assumption and assuming that $\{\|\psi_{\boldsymbol{k}}\|\}_{\boldsymbol{k} \in \mathbb{Z}^d}$ is a bounded sequence, finding equilibrium solutions of (2) in H is equivalent to finding solutions of (5) in $X^{\boldsymbol{s}}$.

Proof. Assume that u is an equilibrium solution of (2) in H. Since $u \in H$, it is given by (15) with $c_{\boldsymbol{k}} := \frac{\langle u, \psi_{\boldsymbol{k}} \rangle}{\|\psi_{\boldsymbol{k}}\|^2}$. Since by assumption $c = \{c_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{Z}^d} \in X^{\boldsymbol{s}}$, then from the construction in Section 1 it is a solution of (5).

For the reciprocal, assume that $c \in X^{\boldsymbol{s}}$ a solution of (5). Since the sequence $\{\|\psi_{\boldsymbol{k}}\|\}_{\boldsymbol{k} \in \mathbb{Z}^d}$ is bounded, we can assume without loss of generality that $\{\psi_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{Z}^d}$ is an orthonormal basis. We have that $\omega_{\boldsymbol{k}}^{\boldsymbol{s}} |c_{\boldsymbol{k}}| \leq \|c\|_{\boldsymbol{s}} < \infty$, which implies that

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^d} |c_{\boldsymbol{k}}|^2 \leq \|c\|_{\boldsymbol{s}}^2 \sum_{\boldsymbol{k} \in \mathbb{Z}^d} (1/\omega_{\boldsymbol{k}}^{\boldsymbol{s}})^2 \leq \|c\|_{\boldsymbol{s}}^2 \frac{\alpha_{\boldsymbol{0}}^{(2)}}{\omega_{\boldsymbol{0}}^{\boldsymbol{s}}} < \infty,$$

where the second inequality follows from Lemma 2.1. Therefore the series (15) converges and we can use it to define u. Combining Lemma 2.1 and the fact that c is a solution of (5), there exists a positive constant D such that

$$|\mu_{\boldsymbol{k}} c_{\boldsymbol{k}}| = \left| \sum_{n=2}^{p} q_n \sum_{\substack{\boldsymbol{k}^1+\cdots+\boldsymbol{k}^n=\boldsymbol{k} \\ \boldsymbol{k}^j \in \mathbb{Z}^d}} c_{\boldsymbol{k}^1} \cdots c_{\boldsymbol{k}^n} \right| \leq \frac{D}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}},$$

for every $\boldsymbol{k} \in \mathbb{Z}^d$. We then have that

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^d} |\mu_{\boldsymbol{k}} c_{\boldsymbol{k}}|^2 = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \left| \sum_{n=2}^{p} q_n \sum_{\substack{\boldsymbol{k}^1+\cdots+\boldsymbol{k}^n=\boldsymbol{k} \\ \boldsymbol{k}^j \in \mathbb{Z}^d}} c_{\boldsymbol{k}^1} \cdots c_{\boldsymbol{k}^n} \right|^2 \leq D^2 \sum_{\boldsymbol{k} \in \mathbb{Z}^d} (1/\omega_{\boldsymbol{k}}^{\boldsymbol{s}})^2 < \infty.$$

This implies that

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^d} \mu_{\boldsymbol{k}} c_{\boldsymbol{k}} \psi_{\boldsymbol{k}}$$

and

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^d} \left(\sum_{n=2}^{p} q_n \sum_{\substack{\boldsymbol{k}^1+\cdots+\boldsymbol{k}^n=\boldsymbol{k} \\ \boldsymbol{k}^j \in \mathbb{Z}^d}} c_{\boldsymbol{k}^1} \cdots c_{\boldsymbol{k}^n} \right) \psi_{\boldsymbol{k}}$$

converge, and therefore u is in the domain of L. Therefore, from the construction in Section 1, u is obviously a equilibrium solution of (2) in H. ■

From now on, we assume that finding zeros of (5) is equivalent to finding equilibria of (2). To study problem (5) in the context of one dimensional domains, the notion of validated continuation was introduced in [3]. Using ideas from this method, we now introduce a rigorous continuation method to prove existence and local uniqueness of equilibria of PDEs defined on d-dimensional domains. The basic idea of the method is to find first a numerical approximation for a zero of (5), then use this approximation to transform (5) into an equivalent fixed point problem, and finally use this fixed point problem to prove the existence and local uniqueness of an equilibrium in a small neighborhood of the initially computed approximation. In order to compute the initial numerical approximation we first need to reduce the infinite dimensional problem (5) to a finite dimensional one. This is obtained by means of a Galerkin projection.

Given $x = \{x_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{Z}^d}$ we denote its *finite part* of size $\boldsymbol{m}$ and its corresponding *infinite part* respectively by $x_{\boldsymbol{F_m}} := \{x_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \boldsymbol{F_m}}$ and $x_{\boldsymbol{I_m}} := \{x_{\boldsymbol{k}}\}_{\boldsymbol{k} \notin \boldsymbol{F_m}}$. Now consider a *Galerkin projection* of (5) of dimension $\boldsymbol{m}$ given by $f^{(\boldsymbol{m})} := \{f_{\boldsymbol{k}}^{(\boldsymbol{m})}\}_{\boldsymbol{k} \in \boldsymbol{F_m}}$, where $f_{\boldsymbol{k}}^{(\boldsymbol{m})} : \mathbb{R}^{m_1 \cdots m_d} \times \mathbb{R} \to \mathbb{R}$, is given by

$$f_{\boldsymbol{k}}^{(\boldsymbol{m})}(x_{\boldsymbol{F_m}}, \lambda) := f_{\boldsymbol{k}}((x_{\boldsymbol{F_m}}, 0_{\boldsymbol{I_m}}), \lambda) = \mu_{\boldsymbol{k}} c_{\boldsymbol{k}} + \sum_{n=2}^{p} q_n \sum_{\substack{\boldsymbol{k}^1+\cdots+\boldsymbol{k}^n=\boldsymbol{k} \\ \boldsymbol{k}^j \in \boldsymbol{F_m}}} c_{\boldsymbol{k}^1} \cdots c_{\boldsymbol{k}^n}, \tag{16}$$

for $\boldsymbol{k} \in \boldsymbol{F_m}$. Now suppose that at the parameter value λ_0, we numerically find $\bar{x}_{\boldsymbol{F_m}}$ such that $f^{(\boldsymbol{m})}(\bar{x}_{\boldsymbol{F_m}}, \lambda_0) \approx 0$. Defining $\bar{x} := (\bar{x}_{\boldsymbol{F_m}}, 0_{\boldsymbol{I_m}}) \in X^{\boldsymbol{s}}$ we assume that $f(\bar{x}, \lambda_0) \approx 0$ and use $\bar{x}$ to define a fixed point problem equivalent to (5). For this purpose, assume that the Jacobian matrix $Df^{(\boldsymbol{m})}(\bar{x}_{\boldsymbol{F_m}}, \lambda_0)$ is non-singular and let $J_{\boldsymbol{m}}^{-1}$ be an approximation for its inverse. In the applications in this paper we take $J_{\boldsymbol{m}}^{-1}$ to be a numerical approximation for the inverse of $Df^{(\boldsymbol{m})}(\bar{x}_{\boldsymbol{F_m}}, \lambda_0)$, but in principle it could be any approximation. The only requirement is that $J_{\boldsymbol{m}}^{-1}$ is non-singular.

We then define the linear operator J^{-1} on sequence spaces, which acts as an approximation for the inverse of $Df(\bar{x},\lambda_0)$. Where, for $x = \{x_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$, it is defined component-wise by

$$\left[J^{-1}(x)\right]_{\boldsymbol{k}} := \begin{cases} \left[J_{\boldsymbol{m}}^{-1}(x_{\boldsymbol{F_m}})\right]_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \in \boldsymbol{F_m} \\ \mu_{\boldsymbol{k}}^{-1} x_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \notin \boldsymbol{F_m}. \end{cases}$$

Using the above we define

$$T(x) = T_{\lambda_0}(x) := x - J^{-1} f(x,\lambda_0).$$

We want to uniquely enclose fixed points of T into closed balls $B(\bar{x},r)$ in $X^{\boldsymbol{s}}$ centered at $\bar{x}$. One can easily check that the closed ball of radius r in $X^{\boldsymbol{s}}$, centered at the origin, is given by

$$B(r) := B(0,r) = \prod_{\boldsymbol{k}\in\mathbb{Z}^d} \left[-\frac{r}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}, \frac{r}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}\right].$$

The closed ball of radius r centered at $\bar{x}$ is

$$B(\bar{x},r) = \bar{x} + B(r).$$

As proved in Lemma 3.3, to show that T is a contraction mapping, we need bounds $Y_{\boldsymbol{k}}$ and $Z_{\boldsymbol{k}}$ for all $\boldsymbol{k}\in\mathbb{Z}^d$, such that

$$\left|\left[T(\bar{x}) - \bar{x}\right]_{\boldsymbol{k}}\right| \leq Y_{\boldsymbol{k}}, \tag{17}$$

and

$$\sup_{b,c\in B(r)} \left|\left[DT(\bar{x}+b)c\right]_{\boldsymbol{k}}\right| \leq Z_{\boldsymbol{k}}. \tag{18}$$

The following Lemma is very similar to Theorem 2.1 in [7], but the Banach space $X^{\boldsymbol{s}}$ involved in the proof is different. Hence, we decide here to present the proof for sake of completeness.

Lemma 3.3. *Fix the parameter value $\lambda = \lambda_0$. If there exists $r > 0$ such that*

$$\|Y + Z\|_{\boldsymbol{s}} < r, \tag{19}$$

with $Y := \{Y_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ and $Z := \{Z_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$, satisfying (17) and (18), respectively, then there is a unique $\tilde{x}\in B(\bar{x},r)$ such that $f(\tilde{x},\lambda_0) = 0$. Moreover, $\tilde{x}$ is in the interior of $B(\bar{x},r)$.

Proof. For given $\boldsymbol{k}\in\mathbb{Z}^d$ and $x,y\in B(\bar{x},r)$, by the Mean Value Theorem, we have that

$$T_{\boldsymbol{k}}(x) - T_{\boldsymbol{k}}(y) = DT_{\boldsymbol{k}}(z)(x-y)$$

for some $z \in \{tx + (1-t)y \mid t\in[0,1]\} \subset B(\bar{x},r)$. Then

$$\left|T_{\boldsymbol{k}}(x) - T_{\boldsymbol{k}}(y)\right| = \left|DT_{\boldsymbol{k}}(z)\frac{r(x-y)}{\|x-y\|_{\boldsymbol{s}}}\right| \frac{1}{r}\|x-y\|_{\boldsymbol{s}} \leq \frac{Z_{\boldsymbol{k}}}{r}\|x-y\|_{\boldsymbol{s}}, \tag{20}$$

and so

$$\left|T_{\boldsymbol{k}}(x) - \bar{x}_{\boldsymbol{k}}\right| \leq \left|T_{\boldsymbol{k}}(x) - T_{\boldsymbol{k}}(\bar{x})\right| + \left|T_{\boldsymbol{k}}(\bar{x}) - \bar{x}_{\boldsymbol{k}}\right| \leq Z_{\boldsymbol{k}} + Y_{\boldsymbol{k}}.$$

Hence,

$$\|T(x) - \bar{x}\|_{\boldsymbol{s}} = \sup_{\boldsymbol{k}\in\mathbb{Z}^d} \omega_{\boldsymbol{k}}^{\boldsymbol{s}}\left|T_{\boldsymbol{k}}(x) - \bar{x}_{\boldsymbol{k}}\right| \leq \sup_{\boldsymbol{k}\in\mathbb{Z}^d} \omega_{\boldsymbol{k}}^{\boldsymbol{s}}\left|Z_{\boldsymbol{k}} + Y_{\boldsymbol{k}}\right| = \|Y+Z\|_{\boldsymbol{s}} < r. \tag{21}$$

This implies that $T(x)\in B(\bar{x},r)$ and hence that T maps $B(\bar{x},r)$ into itself. From (20) we also get that

$$\|T(x) - T(y)\|_{\boldsymbol{s}} = \sup_{\boldsymbol{k}\in\mathbb{Z}^d} \omega_{\boldsymbol{k}}^{\boldsymbol{s}}\left|T_{\boldsymbol{k}}(x) - T_{\boldsymbol{k}}(x)\right| \leq \left(\|Z\|_{\boldsymbol{s}}/r\right)\|x-y\|_{\boldsymbol{s}},$$

and since $Y_{\boldsymbol{k}} \geq 0$ and $Z_{\boldsymbol{k}} \geq 0$, it follows that $\|Z\|_{\boldsymbol{s}} \leq \|Y+Z\|_{\boldsymbol{s}} < r$. We therefore have that the Lipschitz constant of T on $B(\bar{x}, r)$ can be estimated above by $\|Z\|_{\boldsymbol{s}}/r < 1$, and so T is a contraction mapping. Since the operator J^{-1} is invertible, zeros of f correspond to fixed points of T. An application of the Banach Fixed Point Theorem yields the existence of a unique $\tilde{x} \in B(\bar{x}, r)$ such that $T(\tilde{x}) = \tilde{x}$ or equivalently that $f(\tilde{x}, \lambda_0) = 0$. By (21), $\tilde{x}$ is in the interior of $B(\bar{x}, r)$. ■

In order to compute the upper bounds $Y_{\boldsymbol{k}}$ and $Z_{\boldsymbol{k}}$ we choose $\boldsymbol{M} \in \mathbb{N}^d$ such that $\boldsymbol{M} \geq p(\boldsymbol{m}-1)+1$ component-wise, that is,

$$M_j \geq p(m_j - 1) + 1$$

for all $1 \leq j \leq d$, where p is the degree of the polynomial nonlinearity in (2). We have that $T(\bar{x}) - \bar{x} = -J^{-1} f(\bar{x}, \lambda_0)$. Since $\bar{x}$ is such that $\bar{x}_{\boldsymbol{k}} = 0$ for $\boldsymbol{k} \notin \boldsymbol{F_m}$ we get that $f_{\boldsymbol{k}}(\bar{x}, \lambda_0) = 0$ for every $\boldsymbol{k} \notin \boldsymbol{F_M}$. Hence we define $Y = \{Y_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{Z}^d}$ as

$$Y_{\boldsymbol{k}} := \begin{cases} \left[|J_{\boldsymbol{m}}^{-1} f^{(\boldsymbol{m})}(\bar{x}_{\boldsymbol{F_m}}, \lambda_0)| \right]_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \in \boldsymbol{F_m} \\ |\mu_{\boldsymbol{k}}^{-1} f_{\boldsymbol{k}}(\bar{x}, \lambda_0)|, & \text{if } \boldsymbol{k} \in \boldsymbol{F_M} \setminus \boldsymbol{F_m} \\ 0, & \text{if } \boldsymbol{k} \notin \boldsymbol{F_M} \end{cases}$$

Rather than give general formulas for the upper bounds $Z_{\boldsymbol{k}}$, we show explicitly in Section 4 how to compute them for the case of a cubic nonlinearity. However, Lemma 3.4 shows that, as for $Y_{\boldsymbol{k}}$, we can find a uniform upper bound for all $\boldsymbol{k} \notin \boldsymbol{F_M}$, and hence only need to compute with $Z_{\boldsymbol{k}}$ for $\boldsymbol{k} \in \boldsymbol{F_M}$. In order to define this upper bound, we first derive a general formula for $[DT(\bar{x}+b)c]_{\boldsymbol{k}}$ with $\boldsymbol{k} \notin \boldsymbol{F_m}$. Using the notation

$$\left[(\bar{x}+b)^{n-1} * c \right]_{\boldsymbol{k}} := \sum_{\substack{\boldsymbol{k}^1 + \cdots + \boldsymbol{k}^n = \boldsymbol{k} \\ \boldsymbol{k}^j \in \mathbb{Z}^d}} (\bar{x}+b)_{\boldsymbol{k}^1} \cdots (\bar{x}+b)_{\boldsymbol{k}^{n-1}} c_{\boldsymbol{k}^n}$$

we have that

$$\left[Df(\bar{x}+b, \lambda_0)c \right]_{\boldsymbol{k}} = \mu_{\boldsymbol{k}} c_{\boldsymbol{k}} + \sum_{n=2}^{p} n q_n \left[(\bar{x}+b)^{n-1} * c \right]_{\boldsymbol{k}},$$

where $\mu_{\boldsymbol{k}} = \mu_{\boldsymbol{k}}(\lambda_0)$. Now assuming that $\boldsymbol{k} \notin \boldsymbol{F_m}$ and setting $b = ru$ and $c = rv$, for $u, v \in B(1)$, we get, by the Binomial Theorem, that

$$\begin{aligned} \left[DT(\bar{x}+b)c \right]_{\boldsymbol{k}} &= -\frac{1}{\mu_{\boldsymbol{k}}} \sum_{n=2}^{p} n r q_n \left[(\bar{x}+ru)^{n-1} * v \right]_{\boldsymbol{k}} \\ &= -\frac{1}{\mu_{\boldsymbol{k}}} \sum_{n=2}^{p} n r q_n \left(\sum_{j=0}^{n-1} \binom{n-1}{j} r^j \left[\bar{x}^{n-1-j} * u^j * v \right]_{\boldsymbol{k}} \right). \end{aligned} \tag{22}$$

Lemma 3.4. *Assume there is a uniform lower bound*

$$\tilde{\mu}_{\boldsymbol{M}} \leq |\mu_{\boldsymbol{k}}|, \quad \text{for all } \boldsymbol{k} \notin \boldsymbol{F_M}. \tag{23}$$

Then we can find a uniform upper bound $\tilde{Z}_{\boldsymbol{M}}$*, independent of* $\boldsymbol{k}$*, such that*

$$\sup_{b,c \in B(r)} \left| \left[DT(\bar{x}+b)c \right]_{\boldsymbol{k}} \right| \leq \frac{r}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \tilde{Z}_{\boldsymbol{M}} \quad \text{for all } \boldsymbol{k} \notin \boldsymbol{F_M}.$$

Proof. Notice that

$$\left| \bar{x}_{\boldsymbol{k}} \right| \leq \frac{\|\bar{x}\|_{\boldsymbol{s}}}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}$$

for all $\boldsymbol{k} \in \mathbb{Z}^d$. Applying Lemma 2.4 to equation (22), for $\boldsymbol{k} \notin \boldsymbol{F_M}$, we get

$$\begin{aligned}\left|\left[DT(\bar{x}+b)c\right]_{\boldsymbol{k}}\right| &\leq \frac{r}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \frac{1}{|\mu_{\boldsymbol{k}}|} \sum_{n=2}^{p} n|q_n| \left(\sum_{j=0}^{n-1} \binom{n-1}{j} \|\bar{x}\|_{\boldsymbol{s}}^{n-1-j} \tilde{\alpha}_{\boldsymbol{M}}^{(n)} r^j \right) \\ &\leq \frac{r}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \frac{1}{\tilde{\mu}_{\boldsymbol{M}}} \sum_{n=2}^{p} n|q_n| \left(\sum_{j=0}^{n-1} \binom{n-1}{j} \|\bar{x}\|_{\boldsymbol{s}}^{n-1-j} \tilde{\alpha}_{\boldsymbol{M}}^{(n)} r^j \right).\end{aligned}$$

Defining

$$\tilde{Z}_{\boldsymbol{M}} := \frac{1}{\tilde{\mu}_{\boldsymbol{M}}} \sum_{n=2}^{p} n|q_n| \left(\sum_{j=0}^{n-1} \binom{n-1}{j} \|\bar{x}\|_{\boldsymbol{s}}^{n-1-j} \tilde{\alpha}_{\boldsymbol{M}}^{(n)} r^j \right) \tag{24}$$

we get the result. ∎

Using the above we define for $\{Z_{\boldsymbol{k}}\}_{\boldsymbol{k} \notin \boldsymbol{F_M}}$ by

$$Z_{\boldsymbol{k}} := \frac{r}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \tilde{Z}_{\boldsymbol{M}}. \tag{25}$$

Notice that $\tilde{Z}_{\boldsymbol{M}}$ is a polynomial in r, independent of $\boldsymbol{k}$. To define $\{Z_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \boldsymbol{F_M}}$, which are also polynomials in r, we need to compute upper bounds for $\left|\left[DT(\bar{x}+b)c\right]_{\boldsymbol{k}}\right|$. This is done for the case of a cubic nonlinearity in Section 4. In order to verify the existence of a radius r satisfying the hypothesis (19), we introduce the following polynomials.

Definition 3.5. We define the *finite radii polynomials* $\{p_{\boldsymbol{k}}(r)\}_{\boldsymbol{k} \in \boldsymbol{F_M}}$ by

$$p_{\boldsymbol{k}}(r) := Y_{\boldsymbol{k}} + Z_{\boldsymbol{k}} - \frac{r}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}}, \tag{26}$$

and the *tail radii polynomial* by

$$\tilde{p}_{\boldsymbol{M}}(r) := \tilde{Z}_{\boldsymbol{M}} - 1. \tag{27}$$

Corollary 3.6. *Assume that condition (23) in Lemma 3.4 is satisfied and consider the radii polynomials* $\{p_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \boldsymbol{F_M}}$ *and* $\tilde{p}_{\boldsymbol{M}}$ *given by (26) and (27), respectively. If there exists* $r > 0$ *such that* $p_{\boldsymbol{k}}(r) < 0$ *for all* $\boldsymbol{k} \in \boldsymbol{F_M}$ *and* $\tilde{p}_{\boldsymbol{M}}(r) < 0$*, then there is a unique* $\tilde{x} \in B(\bar{x}, r)$ *such that* $f(\tilde{x}, \lambda_0) = 0$*. Moreover,* $\tilde{x}$ *is in the interior of* $B(\bar{x}, r)$*.*

Proof. For $\boldsymbol{k} \in \boldsymbol{F_M}$, notice that $p_{\boldsymbol{k}}(r) < 0$ implies that

$$\omega_{\boldsymbol{k}}^{\boldsymbol{s}} \left|Y_{\boldsymbol{k}} + Z_{\boldsymbol{k}}\right| < r.$$

For $\boldsymbol{k} \notin \boldsymbol{F_M}$, since $Y_{\boldsymbol{k}} = 0$ and $\tilde{p}_{\boldsymbol{M}}(r) < 0$, we get that

$$\omega_{\boldsymbol{k}}^{\boldsymbol{s}} \left|Y_{\boldsymbol{k}} + Z_{\boldsymbol{k}}\right| = \omega_{\boldsymbol{k}}^{\boldsymbol{s}} Z_{\boldsymbol{k}} = r\tilde{Z}_{\boldsymbol{M}} < r.$$

Therefore we have

$$\|Y + Z\|_{\boldsymbol{s}} = \sup_{\boldsymbol{k} \in \mathbb{Z}^d} \omega_{\boldsymbol{k}}^{\boldsymbol{s}} \left|Y_{\boldsymbol{k}} + Z_{\boldsymbol{k}}\right| = \max \left\{ \max_{\boldsymbol{k} \in \boldsymbol{F_M}} \left\{ \omega_{\boldsymbol{k}}^{\boldsymbol{s}} \left|Y_{\boldsymbol{k}} + Z_{\boldsymbol{k}}\right| \right\}, \; r\tilde{Z}_{\boldsymbol{M}} \right\} < r.$$

The result then follows from Lemma 3.3. ∎

Rigorous continuation is based on a classical predictor-corrector continuation algorithm [12]: given, within a prescribed tolerance, a solution u_0 at parameter value λ_0, the predictor step produces an approximate equilibrium $\tilde{u}_1$ at nearby parameter value λ_1, and the corrector step, often based on a Newton-like operator, takes $\tilde{u}_1$ as its input and produces, once again within the prescribed tolerance, a solution u_1 at λ_1. Hence, at every step of the continuation algorithm, we build the radii polynomials defined by (26) and (27) and look for the existence of $r > 0$ such that $p_{\boldsymbol{k}}(r) < 0$ for all $\boldsymbol{k} \in \boldsymbol{F_M}$ and $\tilde{p}_{\boldsymbol{M}}(r) < 0$. If we are successful at a given step, we obtain a proof of existence and local uniqueness of a true equilibrium solution for the original PDE (1), and then we continue to the next step. It is worth pointing out that the computation of the solutions and of the radius r are done using standard numerical methods. Only the computation of the coefficients of the radii polynomials and the check of the polynomial inequalities is made rigorous by using interval arithmetic. This procedure hence yields a computer-assisted proof of existence of solutions. We call this procedure *rigorous continuation* for equilibria of PDEs defined on d-dimensional spatial domains. It is important to note that the main difficulty in the construction of the radii polynomials is to compute the upper bounds $Z_{\boldsymbol{k}}$.

4 Radii Polynomials for Cubic Nonlinearity

In this section we derive the formulas for the radii polynomials for the case of a cubic nonlinearity, that is, for f of the form

$$f_{\boldsymbol{k}}(x,\lambda) := \mu_{\boldsymbol{k}} x_{\boldsymbol{k}} \;+\; q_3 \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^j \in \mathbb{Z}^d}} x_{\boldsymbol{k}^1} x_{\boldsymbol{k}^2} x_{\boldsymbol{k}^3}. \tag{28}$$

In order to compute $Z_{\boldsymbol{k}}$ it is convenient to denote $\tilde{J}_{\boldsymbol{m}} := Df^{(\boldsymbol{m})}(\bar{x}_{\boldsymbol{F_m}}, \lambda_0)$ and introduce the operator

$$\left[\tilde{J}(x)\right]_{\boldsymbol{k}} := \begin{cases} \left[\tilde{J}_{\boldsymbol{m}}(x_{\boldsymbol{F_m}})\right]_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \in \boldsymbol{F_m} \\ \mu_{\boldsymbol{k}} x_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \notin \boldsymbol{F_m}, \end{cases}$$

which acts as an approximate inverse for the operator J^{-1}. We then split $DT(\bar{x}+b)c$ as follows

$$DT(\bar{x}+b)c = \left(I - J^{-1}\tilde{J}\right)c - J^{-1}\left(Df(\bar{x}+b,\lambda_0) - \tilde{J}\right)c, \tag{29}$$

where the first term is very small for $\boldsymbol{k} \in \boldsymbol{F_m}$, and is zero for $\boldsymbol{k} \notin \boldsymbol{F_m}$. For $\boldsymbol{k} \in \boldsymbol{F_m}$ we have the bounds

$$\left|\left[\left(I - J^{-1}\tilde{J}\right)c\right]_{\boldsymbol{k}}\right| \le r\left[\left|I - J_{\boldsymbol{m}}^{-1} Df^{(\boldsymbol{m})}(\bar{x}_{\boldsymbol{F_m}},\lambda_0)\right| \omega_{\boldsymbol{F_m}}^{-\boldsymbol{s}}\right]_{\boldsymbol{k}} =: r Z_{\boldsymbol{k}}^{(0)},$$

where $\omega_{\boldsymbol{F_m}}^{-\boldsymbol{s}} := \{1/\omega_{\boldsymbol{k}}^{\boldsymbol{s}}\}_{\boldsymbol{k}\in\boldsymbol{F_m}}$, and $|\cdot|$ represents component-wise absolute values. As for the second term in (29), we have that

$$\left[Df(\bar{x}+b,\lambda_0)c\right]_{\boldsymbol{k}} = \mu_{\boldsymbol{k}} c_{\boldsymbol{k}} + 3q_3\left[\bar{x}^2 * c + 2\bar{x} * b * c + b^2 * c\right]_{\boldsymbol{k}},$$

and

$$\left[\tilde{J}c\right]_{\boldsymbol{k}} = \begin{cases} \mu_{\boldsymbol{k}} c_{\boldsymbol{k}} \;+\; 3q_3 \displaystyle\sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^j \in \boldsymbol{F_m}}} \bar{x}_{\boldsymbol{k}^1}\bar{x}_{\boldsymbol{k}^2} c_{\boldsymbol{k}^3}, & \text{for } \boldsymbol{k} \in \boldsymbol{F_m} \\ \mu_{\boldsymbol{k}} c_{\boldsymbol{k}}, & \text{for } \boldsymbol{k} \notin \boldsymbol{F_m}. \end{cases}$$

We now consider $u, v \in B(1)$ defined by $b = ru$ and $c = rv$ so that we can expand the expression $\left[\left(Df(\bar{x}+b,\lambda_0) - \tilde{J}\right)c\right]_{\boldsymbol{k}}$ in terms of r as

$$\left[\left(Df(\bar{x}+b,\lambda_0) - \tilde{J}\right)c\right]_{\boldsymbol{k}} = 3q_3\left(C_{\boldsymbol{k}}^{(1)} r + 2C_{\boldsymbol{k}}^{(2)} r^2 + C_{\boldsymbol{k}}^{(3)} r^3\right),$$

where

$$C_{\boldsymbol{k}}^{(1)} := \begin{cases} \displaystyle\sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^1,\boldsymbol{k}^2\in \boldsymbol{F_m},\ \boldsymbol{k}^3\notin \boldsymbol{F_m}}} \bar{x}_{\boldsymbol{k}^1}\bar{x}_{\boldsymbol{k}^2} v_{\boldsymbol{k}^3}, & \text{for } \boldsymbol{k}\in \boldsymbol{F_m} \\ \displaystyle\sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^1,\boldsymbol{k}^2\in \boldsymbol{F_m},\ \boldsymbol{k}^3\in \mathbb{Z}^d}} \bar{x}_{\boldsymbol{k}^1}\bar{x}_{\boldsymbol{k}^2} v_{\boldsymbol{k}^3}, & \text{for } \boldsymbol{k}\notin \boldsymbol{F_m}, \end{cases}$$

$$C_{\boldsymbol{k}}^{(2)} := \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^1\in \boldsymbol{F_m},\ \boldsymbol{k}^2,\boldsymbol{k}^3\in \mathbb{Z}^d}} \bar{x}_{\boldsymbol{k}^1} u_{\boldsymbol{k}^2} v_{\boldsymbol{k}^3}, \quad \text{and} \quad C_{\boldsymbol{k}}^{(3)} := \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^j\in \mathbb{Z}^d}} u_{\boldsymbol{k}^1} u_{\boldsymbol{k}^2} v_{\boldsymbol{k}^3}.$$

We now want to find upper bounds $Z_{\boldsymbol{k}}^{(1)}$, $Z_{\boldsymbol{k}}^{(2)}$ and $Z_{\boldsymbol{k}}^{(3)}$ so that $|C_{\boldsymbol{k}}^{(j)}| \le Z_{\boldsymbol{k}}^{(j)}$, for $j=1,2,3$. Consider the splitting

$$C_{\boldsymbol{k}}^{(1)} = \begin{cases} \displaystyle\sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^1,\boldsymbol{k}^2\in \boldsymbol{F_m},\ \boldsymbol{k}^3\in \boldsymbol{F_M}\setminus \boldsymbol{F_m}}} \bar{x}_{\boldsymbol{k}^1}\bar{x}_{\boldsymbol{k}^2} v_{\boldsymbol{k}^3} + \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^1,\boldsymbol{k}^2\in \boldsymbol{F_m},\ \boldsymbol{k}^3\notin \boldsymbol{F_M}}} \bar{x}_{\boldsymbol{k}^1}\bar{x}_{\boldsymbol{k}^2} v_{\boldsymbol{k}^3}, & \text{for } \boldsymbol{k}\in \boldsymbol{F_m} \\ \displaystyle\sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^1,\boldsymbol{k}^2\in \boldsymbol{F_m},\ \boldsymbol{k}^3\in \boldsymbol{F_M}}} \bar{x}_{\boldsymbol{k}^1}\bar{x}_{\boldsymbol{k}^2} v_{\boldsymbol{k}^3} + \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^1,\boldsymbol{k}^2\in \boldsymbol{F_m},\ \boldsymbol{k}^3\notin \boldsymbol{F_M}}} \bar{x}_{\boldsymbol{k}^1}\bar{x}_{\boldsymbol{k}^2} v_{\boldsymbol{k}^3}, & \text{for } \boldsymbol{k}\notin \boldsymbol{F_m}. \end{cases}$$

Using Lemma 2.2 for $\boldsymbol{k}\in \boldsymbol{F_M}$ we set

$$Z_{\boldsymbol{k}}^{(1)} := \begin{cases} \displaystyle\sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^1,\boldsymbol{k}^2\in \boldsymbol{F_m},\ \boldsymbol{k}^3\in \boldsymbol{F_M}\setminus \boldsymbol{F_m}}} |\bar{x}_{\boldsymbol{k}^1}||\bar{x}_{\boldsymbol{k}^2}|(1/\omega_{\boldsymbol{k}^3}^{\boldsymbol{s}}) + \|\bar{x}\|_{\boldsymbol{s}}^2 \varepsilon_{\boldsymbol{k}}^{(3)}, & \text{for } \boldsymbol{k}\in \boldsymbol{F_m} \\ \displaystyle\sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^1,\boldsymbol{k}^2\in \boldsymbol{F_m},\ \boldsymbol{k}^3\in \boldsymbol{F_M}}} |\bar{x}_{\boldsymbol{k}^1}||\bar{x}_{\boldsymbol{k}^2}|(1/\omega_{\boldsymbol{k}^3}^{\boldsymbol{s}}) + \|\bar{x}\|_{\boldsymbol{s}}^2 \varepsilon_{\boldsymbol{k}}^{(3)}, & \text{for } \boldsymbol{k}\in \boldsymbol{F_M}\setminus \boldsymbol{F_m}. \end{cases}$$

For $C_{\boldsymbol{k}}^{(2)}$ and $C_{\boldsymbol{k}}^{(3)}$ we consider the splittings

$$C_{\boldsymbol{k}}^{(2)} = \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^1\in \boldsymbol{F_m},\ \boldsymbol{k}^2,\boldsymbol{k}^3\in \boldsymbol{F_M}}} \bar{x}_{\boldsymbol{k}^1} u_{\boldsymbol{k}^2} v_{\boldsymbol{k}^3} + \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^1\in \boldsymbol{F_m},\ \{\boldsymbol{k}^2,\boldsymbol{k}^3\}\not\subset \boldsymbol{F_M}}} \bar{x}_{\boldsymbol{k}^1} u_{\boldsymbol{k}^2} v_{\boldsymbol{k}^3},$$

and

$$C_{\boldsymbol{k}}^{(3)} = \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^j\in \boldsymbol{F_M}}} u_{\boldsymbol{k}^1} u_{\boldsymbol{k}^2} v_{\boldsymbol{k}^3} + \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \{\boldsymbol{k}^1,\boldsymbol{k}^2,\boldsymbol{k}^3\}\not\subset \boldsymbol{F_M}}} u_{\boldsymbol{k}^1} u_{\boldsymbol{k}^2} v_{\boldsymbol{k}^3}.$$

We again use Lemma 2.2 for $\boldsymbol{k}\in \boldsymbol{F_M}$ to set

$$Z_{\boldsymbol{k}}^{(2)} := \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^1\in \boldsymbol{F_m},\ \boldsymbol{k}^2,\boldsymbol{k}^3\in \boldsymbol{F_M}}} |\bar{x}_{\boldsymbol{k}^1}|(1/\omega_{\boldsymbol{k}^2}^{\boldsymbol{s}})(1/\omega_{\boldsymbol{k}^3}^{\boldsymbol{s}}) + 2\|\bar{x}\|_{\boldsymbol{s}} \varepsilon_{\boldsymbol{k}}^{(3)}, \quad \text{for } \boldsymbol{k}\in \boldsymbol{F_M},$$

and

$$Z_{\boldsymbol{k}}^{(3)} := \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^j\in \boldsymbol{F_M}}} (1/\omega_{\boldsymbol{k}^1}^{\boldsymbol{s}})(1/\omega_{\boldsymbol{k}^2}^{\boldsymbol{s}})(1/\omega_{\boldsymbol{k}^3}^{\boldsymbol{s}}) + 3\varepsilon_{\boldsymbol{k}}^{(3)}, \quad \text{for } \boldsymbol{k}\in \boldsymbol{F_M}.$$

Finally, using (25) for the case $\boldsymbol{k}\notin \boldsymbol{F_M}$, we have

$$Z_{\boldsymbol{k}} := \begin{cases} 3|q_3| \left[\left| J_{\boldsymbol{m}}^{-1} \right| \left(Z_{\boldsymbol{F_m}}^{(1)} r + 2 Z_{\boldsymbol{F_m}}^{(2)} r^2 + Z_{\boldsymbol{F_m}}^{(3)} r^3 \right) \right]_{\boldsymbol{k}} + Z_{\boldsymbol{k}}^{(0)} r, & \text{for } \boldsymbol{k}\in \boldsymbol{F_m} \\ 3|q_3||\mu_{\boldsymbol{k}}^{-1}| \left(Z_{\boldsymbol{k}}^{(1)} r + 2 Z_{\boldsymbol{k}}^{(2)} r^2 + Z_{\boldsymbol{k}}^{(3)} r^3 \right), & \text{for } \boldsymbol{k}\in \boldsymbol{F_M}\setminus \boldsymbol{F_m} \\ \dfrac{r}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \tilde{Z}_{\boldsymbol{M}}, & \text{for } \boldsymbol{k}\notin \boldsymbol{F_M}, \end{cases}$$

where $\tilde{Z}_{\boldsymbol{M}}$, defined in (24), is given by

$$\tilde{Z}_{\boldsymbol{M}} = \frac{1}{\tilde{\mu}_{\boldsymbol{M}}} 3|q_3| \left(\|\bar{x}\|_{\boldsymbol{s}}^2 \tilde{\alpha}_{\boldsymbol{M}}^{(3)} + 2\|\bar{x}\|_{\boldsymbol{s}} \tilde{\alpha}_{\boldsymbol{M}}^{(3)} r + \tilde{\alpha}_{\boldsymbol{M}}^{(3)} r^2 \right).$$

We then have that the radii polynomials, defined in Definition 3.5, for the general cubic problem (28) are given, for $\boldsymbol{k} \in \boldsymbol{F_m}$, by

$$\begin{aligned} p_{\boldsymbol{k}}(r) \quad &= \quad Y_{\boldsymbol{k}} + \left(Z_{\boldsymbol{k}}^{(0)} + 3|q_3| \left[\left| J_{\boldsymbol{m}}^{-1} \right| Z_{\boldsymbol{F_m}}^{(1)} \right]_{\boldsymbol{k}} - 1/\omega_{\boldsymbol{k}}^{\boldsymbol{s}} \right) r + \\ & \left(6|q_3| \left[\left| J_{\boldsymbol{m}}^{-1} \right| Z_{\boldsymbol{F_m}}^{(2)} \right]_{\boldsymbol{k}} \right) r^2 + \left(3|q_3| \left[\left| J_{\boldsymbol{m}}^{-1} \right| Z_{\boldsymbol{F_m}}^{(3)} \right]_{\boldsymbol{k}} \right) r^3, \end{aligned}$$

for $\boldsymbol{k} \in \boldsymbol{F_M} \setminus \boldsymbol{F_m}$, by

$$p_{\boldsymbol{k}}(r) = Y_{\boldsymbol{k}} + \left(\frac{3|q_3|Z_{\boldsymbol{k}}^{(1)}}{|\mu_{\boldsymbol{k}}|} - \frac{1}{\omega_{\boldsymbol{k}}^{\boldsymbol{s}}} \right) r + \frac{6|q_3|Z_{\boldsymbol{k}}^{(2)}}{|\mu_{\boldsymbol{k}}|} r^2 + \frac{3|q_3|Z_{\boldsymbol{k}}^{(3)}}{|\mu_{\boldsymbol{k}}|} r^3,$$

and finally,

$$\tilde{p}_{\boldsymbol{M}}(r) = \frac{3|q_3|\tilde{\alpha}_{\boldsymbol{M}}^{(3)}}{\tilde{\mu}_{\boldsymbol{M}}} r^2 + \frac{6|q_3|\|\bar{x}\|_{\boldsymbol{s}} \tilde{\alpha}_{\boldsymbol{M}}^{(3)}}{\tilde{\mu}_{\boldsymbol{M}}} r + \frac{3|q_3|\|\bar{x}\|_{\boldsymbol{s}}^2 \tilde{\alpha}_{\boldsymbol{M}}^{(3)}}{\tilde{\mu}_{\boldsymbol{M}}} - 1. \tag{30}$$

Remark 4.1. *Note that $\tilde{p}_{\boldsymbol{M}}$, given by (30), always has two distinct real roots, since its discriminant equals $\frac{12|q_3|\tilde{\alpha}_{\boldsymbol{M}}^{(3)}}{\tilde{\mu}_{\boldsymbol{M}}} > 0$. Hence, the only way we could fail to find a positive r such that $\tilde{p}_{\boldsymbol{M}}(r) < 0$ is if $\frac{3|q_3|\|\bar{x}\|_{\boldsymbol{s}}^2 \tilde{\alpha}_{\boldsymbol{M}}^{(3)}}{\tilde{\mu}_{\boldsymbol{M}}} - 1 \geq 0$. In practice, before starting the rigorous numerical computations of the radii polynomials, we check if*

$$\frac{3|q_3|\|\bar{x}\|_{\boldsymbol{s}}^2 \tilde{\alpha}_{\boldsymbol{M}}^{(3)}}{\tilde{\mu}_{\boldsymbol{M}}} < 1. \tag{31}$$

If condition (31) is not satisfied, we a priori know that the proof fails. Hence, we need to increase the value of $\tilde{\mu}_{\boldsymbol{M}}$, which, as shown in Section 5, can be done by increasing the computational parameter $\boldsymbol{M}$.

5 Applications

In this section we present applications to the Cahn-Hilliard and the Swift-Hohenbergh PDEs defined on two- and three-dimensional domains. For all the examples in this section we arbitrarily choose an interval for the continuation parameter and compute all the solutions bifurcating from the trivial solution along that parameter interval. We then follow each branch of solutions for several steps until the running time of the computations reaches a fixed maximum allowed time. No attempt is made to compute the complete bifurcation diagram for the given PDE, nor to continue the solution branches for larger values of the continuation parameter, since the goal is to show the applicability of the method.

For all the computations in this section we use the projection dimension $\boldsymbol{m}$, the computational parameter $\boldsymbol{M}$, and the decay rate $\boldsymbol{s}$ uniform component-wise, that is, we use $\boldsymbol{m} = (m, \ldots, m)$, $\boldsymbol{M} = (M, \ldots, M)$, and $\boldsymbol{s} = (s, \ldots, s)$. In all the computations we use the projection dimension $m = 8$ along the trivial branch $u \equiv 0$ and $m = 8$ to start the branches bifurcating from this trivial branch. At each step of the continuation algorithm if the proof is successful we proceed to the next step along the branch using the same projection dimension. If, on a given step, the proof fails we increase the projection dimension m by one, recompute the solution at that step and try to prove existence again. We then repeat this process for all the steps in the computations. Using this approach we proved the existence of all solutions presented in this section.

As for the computational parameters M and s, we choose them either arbitrarily or based on experimentation. For the computations in this paper we choose M as the smallest integer such that $M \geq 3m-2$ and that condition (31) is satisfied.

5.1 Cahn-Hilliard Equation

In this section we apply rigorous continuation to the Cahn-Hilliard equation

$$\begin{cases} u_t = -\Delta(\varepsilon^2 \Delta u + u - u^3), & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \Delta u}{\partial n} = 0, & \text{on } \partial\Omega \end{cases} \tag{32}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded rectangular domain, $\varepsilon > 0$ models the interaction length, and n denotes the unit outer normal to $\partial\Omega$, that is, we have no-flux boundary conditions for both u and Δu. Equation (32) was introduced in [13, 14, 15] as a model for phase separation in binary alloys. The model is mass preserving, meaning that, for any solution u, the total mass

$$\sigma := \frac{1}{|\Omega|} \int_\Omega u(y,t) dy$$

remains constant for all $t \geq 0$, which introduces the additional parameter σ. The equilibria of (32) are given by the solutions of

$$\begin{cases} \varepsilon^2 \Delta u + u - u^3 = c & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

which introduce yet another parameter

$$c := \frac{1}{|\Omega|} \int_\Omega \left(u(y) - u(y)^3 \right) dy.$$

In this paper we assume that both alloys have equal concentrations, which means that the total mass is equal to zero. We also consider only the case $c = 0$. In this case, studying the equilibria of (32) is equivalent to studying the equilibria of the Allen-Cahn equation [16]

$$\begin{cases} u_t = \varepsilon^2 \Delta u + u - u^3, & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{33}$$

Due to the Neumann boundary conditions, if we consider the domain as

$$\Omega = \prod_{j=1}^{d} [0, \ell_j],$$

we can express the solutions in terms of a cosine basis $\{\psi_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{N}^d}$ given by

$$\psi_{\boldsymbol{k}}(y) := \prod_{j=1}^{d} \cos(k_j L_j y_j),$$

where $L_j = \pi/\ell_j$, for $j = 1, \ldots, d$. Notice that we only need to consider the basis elements for $\boldsymbol{k} \geq \boldsymbol{0}$. However, if we use the expansion

$$u = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} a_{\boldsymbol{k}} \psi_{\boldsymbol{k}}$$

with the assumption that $a_{|\boldsymbol{k}|} = a_{\boldsymbol{k}}$ for $\boldsymbol{k} \in \mathbb{Z}^d$, then the expansion of (33) takes the form

$$f_{\boldsymbol{k}}(a,\lambda) := \mu_{\boldsymbol{k}} a_{\boldsymbol{k}} \quad - \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^j \in \mathbb{Z}^d}} a_{\boldsymbol{k}^1} a_{\boldsymbol{k}^2} a_{\boldsymbol{k}^3},$$

where

$$\mu_{\boldsymbol{k}} = 1 - \frac{1}{\lambda}\Big(k_1^2 L_1^2 + \cdots + k_d^2 L_d^2\Big),$$

with $\lambda = 1/\varepsilon^2$, and we have that $f_{|\boldsymbol{k}|} = f_{\boldsymbol{k}}$, for all $\boldsymbol{k} \in \mathbb{Z}^d$. Therefore, we only need to solve $f_{\boldsymbol{k}} = 0$ for $\boldsymbol{k} \geq \boldsymbol{0}$. The remaining quantity needed to construct the radii polynomials of Section 4 is $\tilde{\mu}_{\boldsymbol{M}}$ satisfying (23).

Lemma 5.1 (Construction of $\tilde{\mu}_{\boldsymbol{M}} > 0$)**.** *Assuming that*

$$\min_{1\le j\le d} \left\{M_j^2 L_j^2\right\} > \lambda, \tag{34}$$

and defining

$$\tilde{\mu}_{\boldsymbol{M}} := \frac{1}{\lambda} \min_{1\le j\le d} \left\{M_j^2 L_j^2\right\} - 1 > 0, \tag{35}$$

we have that

$$|\mu_{\boldsymbol{k}}| \geq \tilde{\mu}_{\boldsymbol{M}}, \quad \textit{for all } \boldsymbol{k} \notin \boldsymbol{F}_{\boldsymbol{M}}.$$

Proof. Given $\boldsymbol{k} \notin \boldsymbol{F}_{\boldsymbol{M}}$, there exists $1 \le j_0 \le d$ such that $k_{j_0} \geq M_{j_0}$, then

$$\frac{1}{\lambda}\Big(k_1^2 L_1^2 + \cdots + k_d^2 L_d^2\Big) \geq \frac{1}{\lambda}\Big(M_{j_0}^2 L_{j_0}^2\Big) \geq \frac{1}{\lambda}\min_{1\le j\le d}\left\{M_j^2 L_j^2\right\} > 1,$$

where the last inequality follows from (34). Therefore we have

$$|\mu_{\boldsymbol{k}}| = \frac{1}{\lambda}\Big(k_1^2 L_1^2 + \cdots + k_d^2 L_d^2\Big) - 1 \geq \frac{1}{\lambda}\min_{1\le j\le d}\left\{M_j^2 L_j^2\right\} - 1 = \tilde{\mu}_{\boldsymbol{M}}.$$ ■

Remark 5.2. *Notice that we can always ensure that condition (34) is satisfied by taking* $\boldsymbol{M}$ *large enough.*

We present some results for the Cahn-Hilliard equation in a two-dimensional rectangle in Figure 1 and Figure 2, and in a three-dimensional rectangle in Figure 3 and Figure 4.

5.2 Swift-Hohenberg Equation

In this section we consider the Swift-Hohenberg equation

$$u_t = \nu u - (1+\Delta)^2 u - u^3 \tag{36}$$

with periodic boundary conditions on a rectangular bounded domain $\Omega \subset \mathbb{R}^d$. Equation (36) was introduced in [17] to describe the onset of Rayleigh-Bénard convection, and is widely used as a model for pattern formation. The parameter $\nu > 0$ is the reduced Rayleigh number. In addition to the periodic boundary conditions, we assume the following symmetry conditions

$$u(y,t) = u(|y|,t), \tag{37}$$

for all $x \in \mathbb{R}^d$, where $|y| := (|y_1|, \ldots, |y_d|)$. This means that we are looking for solutions of (36) that are even and periodic in each of the space variables. Due to this symmetry and the boundary conditions, if we take the domain as

$$\Omega = \prod_{j=1}^{d} [0, \ell_j],$$

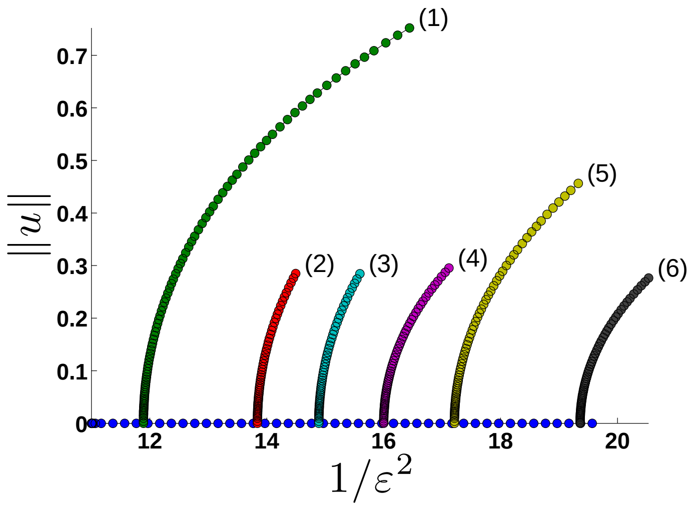

Figure 1: Some branches of equilibria for the Cahn-Hilliard equation in the 2D domain $\Omega = [0, \pi] \times [0, \pi/1.1]$. We refer to these branches as branch 1 through branch 6 according to the labels above. For $\lambda = 1/\epsilon^2$ in the interval $[11, 19.5]$ all the bifurcations from the trivial solution are computed. They occur at $\lambda \approx 11.89$, 13.84, 14.89, 16, 17.21, and 19.36. The proof was successful for all the points in each of the branches in the plot with $s = 2$.

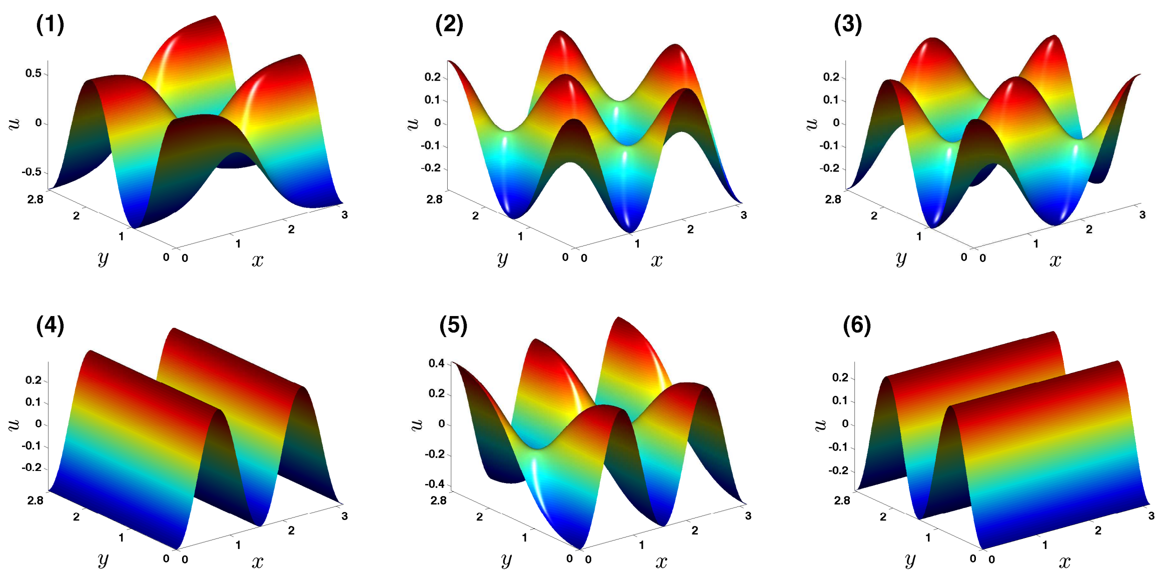

Figure 2: Solutions for the Cahn-Hilliard equation in 2D. Plotted are the solutions corresponding to the last point of the respective branches in Figure 1. Plot **(1)** corresponds to the branch 1 and is computed using $m = 28$ and $M = 1090$; **(2)** corresponds to the branch 2 and is computed using $m = 38$ and $M = 1352$; **(3)** corresponds to the branch 3 and is computed using $m = 38$ and $M = 1400$; **(4)** corresponds to the branch 4 and is computed using $m = 13$ and $M = 1355$; **(5)** corresponds to the branch 5 and is computed using $m = 15$ and $M = 1111$; and **(6)** corresponds to the branch 6 and is computed using $m = 13$ and $M = 1387$.

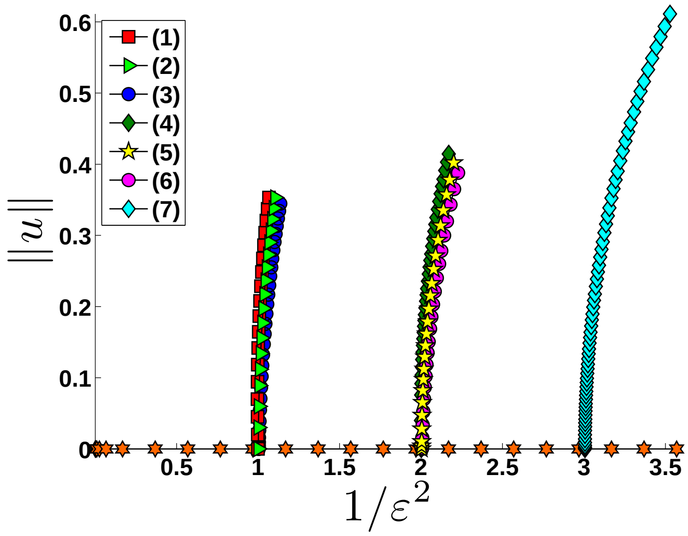

Figure 3: Some branches of equilibria for the Cahn-Hilliard equation in the 3D domain $\Omega = [0,\pi] \times [0,\pi/1.001] \times [0,\pi/1.002]$. We refer to these branches as branch 1 through branch 7 according to the labels above. For $\lambda = 1/\epsilon^2$ in the interval $(0, 3.5]$ all the bifurcations from the trivial solution are computed. They occur at $\lambda \approx 1$, 1.002, 1.004, 2.002, 2.004, 2.006, and 3.006. For all the points in each of the branches in the plot, the proof was successful using $s = 2$.

we can expand the solutions using a cosine basis $\{\psi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}^d}$ given by

$$\psi_{\boldsymbol{k}}(y) := \prod_{j=1}^{d} \cos(k_j L_j y_j),$$

where $L_j = 2\pi/\ell_j$, for $j = 1,\ldots,d$. As in the previous section, we only need to consider the basis elements for $\boldsymbol{k} \geq \boldsymbol{0}$. However, if we use the expansion

$$u = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a_{\boldsymbol{k}} \psi_{\boldsymbol{k}}$$

with the assumption that $a_{|\boldsymbol{k}|} = a_{\boldsymbol{k}}$ for $\boldsymbol{k} \in \mathbb{Z}^d$, then the expansion of (36) takes the form

$$f_{\boldsymbol{k}}(a,\lambda) := \mu_{\boldsymbol{k}} a_{\boldsymbol{k}} \quad - \sum_{\substack{\boldsymbol{k}^1+\boldsymbol{k}^2+\boldsymbol{k}^3=\boldsymbol{k} \\ \boldsymbol{k}^j\in\mathbb{Z}^d}} a_{\boldsymbol{k}^1} a_{\boldsymbol{k}^2} a_{\boldsymbol{k}^3},$$

where

$$\mu_{\boldsymbol{k}} = \lambda - \left[1 - \left(k_1^2 L_1^2 + \cdots + k_d^2 L_d^2\right)\right]^2,$$

with $\lambda = \nu$, and $f_{|\boldsymbol{k}|} = f_{\boldsymbol{k}}$, for all $\boldsymbol{k} \in \mathbb{Z}^d$. Therefore, we only need to solve $f_{\boldsymbol{k}} = 0$ for $\boldsymbol{k} \geq \boldsymbol{0}$. As in Section 5.1, we need to compute $\tilde{\mu}_{\boldsymbol{M}}$ satisfying (23).

Lemma 5.3 (Construction of $\tilde{\mu}_{\boldsymbol{M}} > 0$)**.** *Assuming that*

$$\min_{1\leq j\leq d} \left\{M_j^2 L_j^2\right\} > 1 + \sqrt{\lambda}, \tag{38}$$

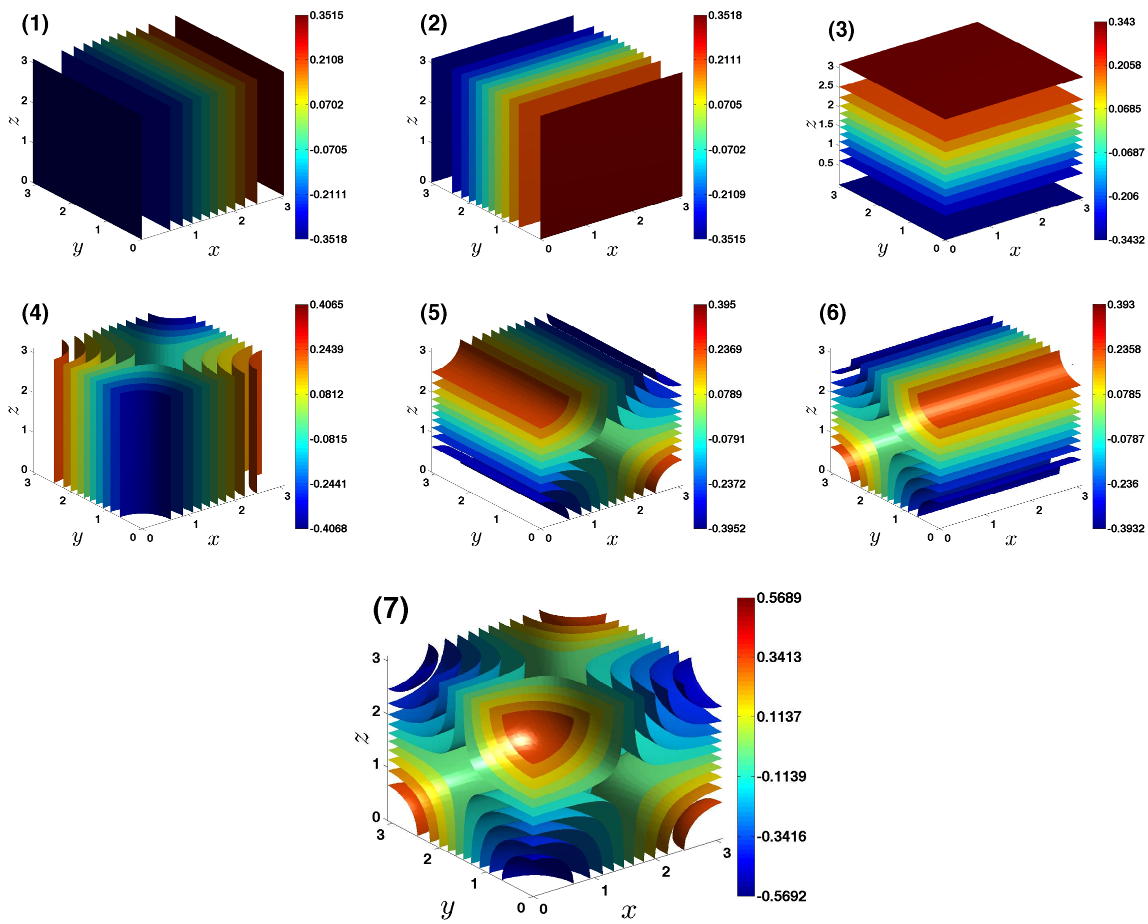

Figure 4: Solutions for the Cahn-Hilliard equation in 3D. Plotted are isosurfaces of the solutions corresponding to the last point of the respective branches in Figure 3. Plot **(1)** corresponds to the branch 1 and is computed using $m = 8$ and $M = 218$; **(2)** corresponds to the branch 2 and is computed using $m = 8$ and $M = 218$; **(3)** corresponds to the branch 3 and is computed using $m = 8$ and $M = 212$; **(4)** corresponds to the branch 4 and is computed using $m = 8$ and $M = 182$; **(5)** corresponds to the branch 5 and is computed using $m = 8$ and $M = 176$; **(6)** corresponds to the branch 6 and is computed using $m = 8$ and $M = 176$; and **(7)** corresponds to the branch 7 and is computed using $m = 8$ and $M = 170$.

and defining

$$\tilde{\mu}_{\boldsymbol{M}} := \left[\min_{1\le j\le d} \left\{ M_j^2 L_j^2 \right\} - 1 \right]^2 - \lambda > 0, \tag{39}$$

we have that

$$|\mu_{\boldsymbol{k}}| \ge \tilde{\mu}_{\boldsymbol{M}}, \quad \text{for all } \boldsymbol{k} \notin \boldsymbol{F}_{\boldsymbol{M}}.$$

Proof. Given $\boldsymbol{k} \notin \boldsymbol{F}_{\boldsymbol{M}}$, there exists $1 \le j_0 \le d$ such that $k_{j_0} \ge M_{j_0}$, then

$$k_1^2 L_1^2 + \cdots + k_d^2 L_d^2 \ge M_{j_0}^2 L_{j_0}^2 \ge \min_{1\le j\le d} \left\{ M_j^2 L_j^2 \right\} > 1,$$

which implies that

$$\left[\left(k_1^2 L_1^2 + \cdots + k_d^2 L_d^2 \right) - 1 \right]^2 \ge \left[\min_{1\le j\le d} \left\{ M_j^2 L_j^2 \right\} - 1 \right]^2 > \lambda.$$

where the last inequality follows from (38). Therefore we conclude that

$$|\mu_{\boldsymbol{k}}| = \left[1 - \left(k_1^2 L_1^2 + \cdots + k_d^2 L_d^2 \right) \right]^2 - \lambda \ge \left[1 - \min_{1\le j\le d} \left\{ M_j^2 L_j^2 \right\} \right]^2 - \lambda = \tilde{\mu}_{\boldsymbol{M}}. \qquad \blacksquare$$

Remark 5.4. *Notice that, as for the Cahn-Hilliard equation, we can always ensure that condition (38) is satisfied by increasing increasing* $\boldsymbol{M}$*.*

We present some results for the Swift-Hohenberg equation in a two-dimensional rectangle in Figure 5 and Figure 6, and in a three-dimensional rectangle in Figure 7 and Figure 8.

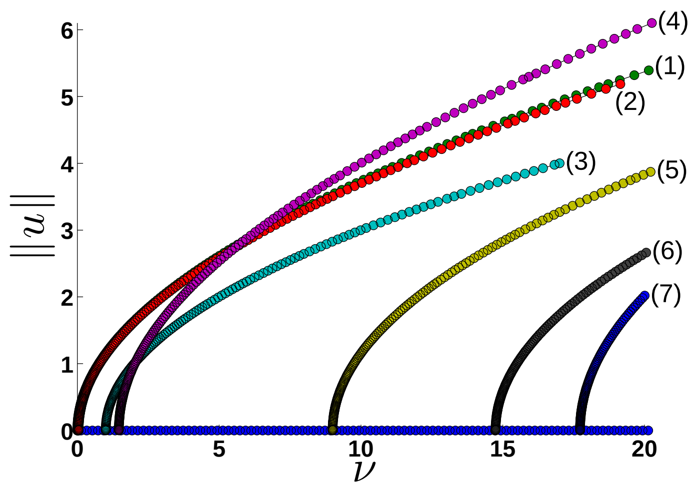

Figure 5: Some branches of equilibria for the Swift-Hohenberg equation in the 2D rectangle $\Omega = [0, 2\pi] \times [0, 2\pi/1.1]$. We refer to these branches as branch 1 through branch 7 according to the labels above. For ν in the interval $[0, 20]$ all the bifurcations from the trivial solution are computed. They occur at $\nu \approx 0$, 0.0441, 1, 1.4641, 9, 14.7456, and 17.7241. The proof was successful for all the points in each of the branches in the plot using $s = 2$.

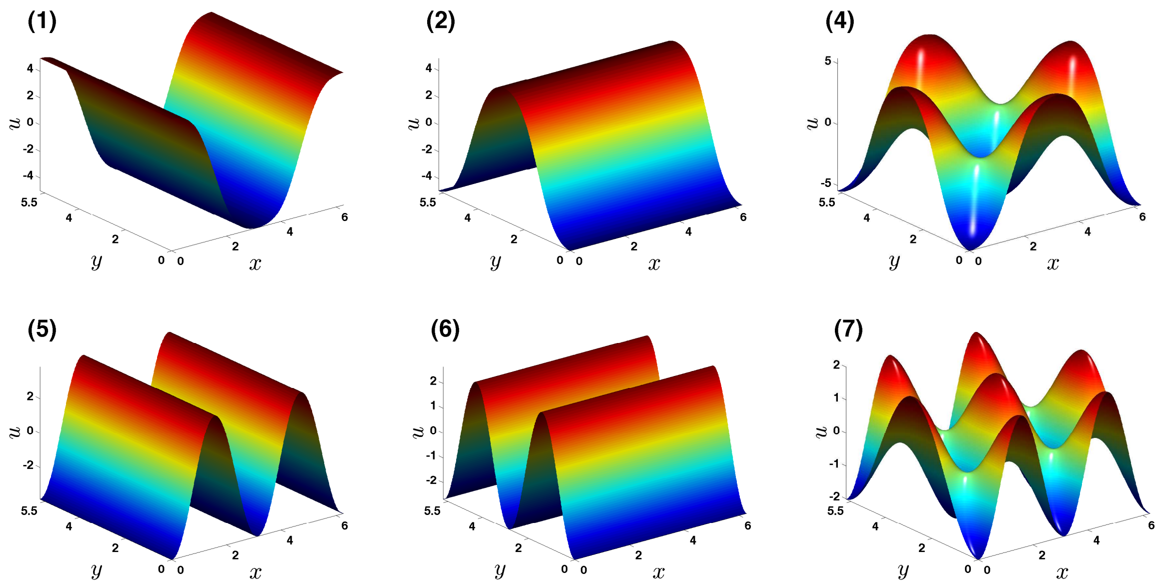

Figure 6: Solutions for the Swift-Hohenberg equation in 2D. Plotted are the solutions corresponding to the last point of the respective branches in Figure 5. Branch 3 corresponds to the trivial solution $u \equiv \sqrt{\nu - 1}$, and hence is not plotted. Plot **(1)** corresponds to the branch 1 and is computed using $m = 24$ and $M = 72$; **(2)** corresponds to the branch 2 and is computed using $m = 8$ and $M = 24$; **(4)** corresponds to the branch 4 and is computed using $m = 8$ and $M = 24$; **(5)** corresponds to the branch 5 and is computed using $m = 8$ and $M = 38$; **(6)** corresponds to the branch 6 and is computed using $m = 8$ and $M = 32$; and **(7)** corresponds to the branch 7 and is computed using $m = 8$ and $M = 24$.

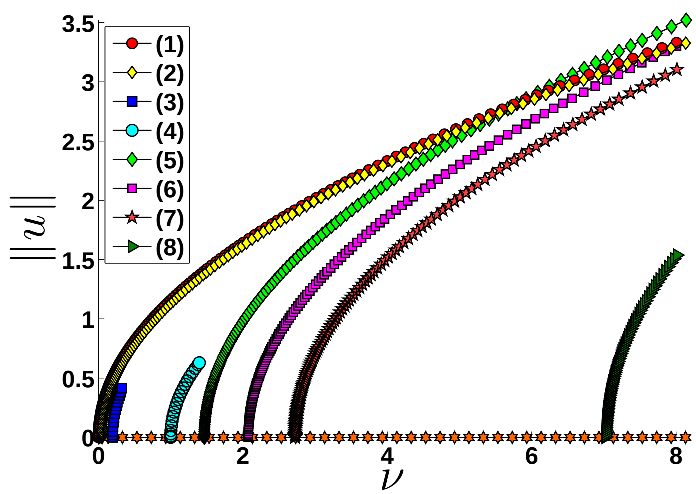

Figure 7: Some branches of equilibria for the Swift-Hohenberg equation in the 3D rectangle $\Omega = [0, 2\pi] \times [0, 2\pi/1.1] \times [0, 2\pi/1.2]$. We refer to these branches as branch 1 through branch 8 according to the labels above. For ν in the interval $[0, 8]$ all the bifurcations from the trivial solution are computed. They occur at $\nu \approx 0$, 0.0441, 0.1936, 1, 1.4641, 2.0736, 2.7225, and 7.0225. The proof was successful for all the points in each of the branches in the plot with $s = 4$.

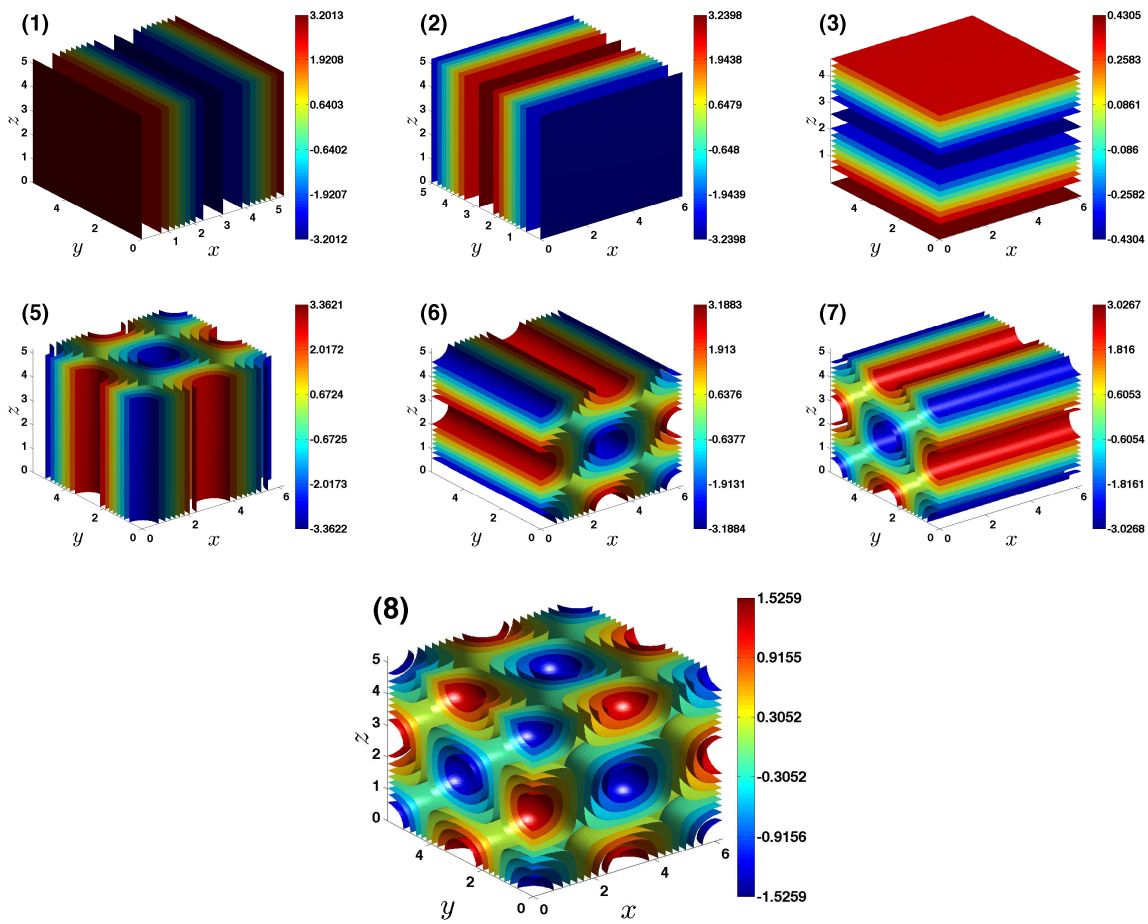

Figure 8: Solutions for the Swift-Hohenberg equation in 3D. Plotted are isosurfaces of the solutions corresponding to the last point of the respective branches in Figure 5. Branch 4 corresponds to the trivial solution $u \equiv \sqrt{\nu - 1}$. Plot **(1)** corresponds to the branch 1 and is computed using $m = 8$ and $M = 48$; **(2)** corresponds to the branch 2 and is computed using $m = 8$ and $M = 48$; **(3)** corresponds to the branch 3 and is computed using $m = 14$ and $M = 42$; **(5)** corresponds to the branch 5 and is computed using $m = 8$ and $M = 36$; **(6)** corresponds to the branch 6 and is computed using $m = 8$ and $M = 36$; **(7)** corresponds to the branch 7 and is computed using $m = 8$ and $M = 34$; and **(8)** corresponds to the branch 8 and is computed using $m = 8$ and $M = 24$.

6 Conclusion

The emphasis of this paper is on the presentation of the analytic estimates and on the presentation of a new rigorous continuation for equilibria of higher-dimensional PDEs. As already mentioned at the beginning of Section 5, no attempts were made to do more extensive computations. We plan next to use our method to analyze the bifurcation structure of the PDEs considered in this paper, as well as other model problems. In addition to that, we propose to apply the theory introduced in [10] to the computation of global smooth branches of equilibria of higher-dimensional PDEs. As a consequence of such a rigorous computation, we would have results about non-existence of secondary bifurcations from the rigorously computed smooth branches of equilibria. To the best of our knowledge this would be the first time that such a method is presented in the context of nonlinear PDEs defined on spatial domains higher than one.

7 Acknowledgements

The authors would like to thank Tom Wanner for helpful discussions.

A One-dimensional Estimates

In this section we present the one-dimensional estimates from [9]. First, let us recall some quantities introduced in [9]. Consider a decay rate $s \geq 2$, a computational parameter $M \geq 6$ and define, for $k \geq 3$,

$$\gamma_k = \gamma_k(s) := 2\left[\frac{k}{k-1}\right]^s + \left[\frac{4\ln(k-2)}{k} + \frac{\pi^2-6}{3}\right]\left[\frac{2}{k}+\frac{1}{2}\right]^{s-2}. \tag{40}$$

Then, for $k \in \mathbb{Z}$, we define $\alpha_k^{(2)} = \alpha_k^{(2)}(s,M)$ by

$$\alpha_k^{(2)} := \begin{cases} 4 + \frac{1}{2^{2s-1}(2s-1)}, & \text{for } k=0 \\ 2\left[2 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{3^{s-1}(s-1)}\right] + \sum_{k_1=1}^{k-1} \frac{k^s}{k_1^s(k-k_1)^s}, & \text{for } 1 \leq k \leq M-1 \\ 2\left[2 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{3^{s-1}(s-1)}\right] + \gamma_k, & \text{for } k \geq M, \end{cases}$$

and for $k < 0$,

$$\alpha_k^{(2)} := \alpha_{|k|}^{(2)}.$$

We also define $\alpha_k^{(n)} = \alpha_k^{(n)}(s,M)$, for $n \geq 3$, by

$$\alpha_k^{(n)} := \begin{cases} \alpha_0^{(n-1)} + 2\sum_{k_1=1}^{M-1} \frac{\alpha_{k_1}^{(n-1)}}{k_1^{2s}} + \frac{2\alpha_M^{(n-1)}}{(M-1)^{2s-1}(2s-1)}, & \text{for } k=0 \\ \sum_{k_1=1}^{M-k-1} \frac{\alpha_{k+k_1}^{(n-1)} k^s}{k_1^s(k+k_1)^s} + \alpha_M^{(n-1)} k^s \left[\frac{1}{(M-k)^s M^s} + \frac{1}{(M-k)^{s-1}(M+1)^s(s-1)}\right] \\ \quad + \alpha_k^{(n-1)} + \sum_{k_1=1}^{k-1} \frac{\alpha_{k_1}^{(n-1)} k^s}{k_1^s(k-k_1)^s} + \alpha_0^{(n-1)} + \sum_{k_1=1}^{M-1} \frac{\alpha_{k_1}^{(n-1)} k^s}{k_1^s(k+k_1)^s} \\ \quad + \frac{\alpha_M^{(n-1)}}{(M-1)^{s-1}(s-1)}, & \text{for } 1 \leq k \leq M-1 \\ \alpha_M^{(n-1)}\left[2 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{3^{s-1}(s-1)} + \frac{1}{(M-1)^{s-1}(s-1)} + \gamma_k\right] \\ \quad + \alpha_0^{(n-1)} + \sum_{k_1=1}^{M-1}\left(\frac{\alpha_{k_1}^{(n-1)}}{k_1^s}\left[1 + \frac{M^s}{(M-k_1)^s}\right]\right), & \text{for } k \geq M \end{cases}$$

and for $k < 0$,

$$\alpha_k^{(n)} := \alpha_{|k|}^{(n)}.$$

Remark A.1. *For any $k \in \mathbb{Z}$ with $|k| \geq M \geq 6$, we have that $\alpha_k^{(n)} \leq \alpha_M^{(n)}$.*

Proof. For $k \geq 6$, the fact that $\frac{\ln(k-1)}{k+1} \leq \frac{\ln(k-2)}{k}$ implies that $\gamma_{k+1}(s) \leq \gamma_k(s)$. By definition of $\alpha_k^{(2)}$, for $|k| \geq M$, one gets that $\alpha_k^{(2)} \leq \alpha_M^{(2)}$. The conclusion follows from the construction of $\alpha_k^{(n)}$, for $|k| \geq M$. ■

We also define the one-dimensional weights

$$\omega_k^s := \begin{cases} 1, & \text{if } k = 0 \\ |k|^s, & \text{if } k \neq 0. \end{cases} \tag{41}$$

The goal is to find asymptotic bounds for infinite convolution sums of the form

$$\sum_{\substack{k_1+\cdots+k_n=k \\ k_j \in \mathbb{Z}}} c_{k_1}^{(1)} \cdots c_{k_n}^{(n)},$$

assuming that the sequences $c^{(j)} = \{c_k^{(j)}\}_{k\in\mathbb{Z}}$ have decay rates of the form

$$\left| c_k^{(j)} \right| \leq \frac{A_j}{\omega_k^s}.$$

First notice that

$$\left| \sum_{\substack{k_1+\cdots+k_n=k \\ k_j \in \mathbb{Z}}} c_{k_1}^{(1)} \cdots c_{k_n}^{(n)} \right| \leq \sum_{\substack{k_1+\cdots+k_n=k \\ k_j \in \mathbb{Z}}} \frac{A_1 \cdots A_n}{\omega_{k_1}^s \cdots \omega_{k_n}^s},$$

and

$$\sum_{\substack{k_1+\cdots+k_n=-k \\ k_j \in \mathbb{Z}}} \frac{1}{\omega_{k_1}^s \cdots \omega_{k_n}^s} = \sum_{\substack{k_1+\cdots+k_n=k \\ k_j \in \mathbb{Z}}} \frac{1}{\omega_{k_1}^s \cdots \omega_{k_n}^s}.$$

Therefore for the rest of this section we just need to consider the cases $k \in \mathbb{N}$, and sums of the form

$$\sum_{\substack{k_1+\cdots+k_n=k \\ k_j \in \mathbb{Z}}} \frac{1}{\omega_{k_1}^s \cdots \omega_{k_n}^s}.$$

Lemma A.2. *For $s \geq 2$ and $k \geq 4$ we have*

$$\sum_{k_1=1}^{k-1} \frac{k^s}{k_1^s (k-k_1)^2} \leq \gamma_k.$$

Proof. First observe that

$$\begin{aligned}
\sum_{k_1=1}^{k-1} \frac{k^s}{k_1^s (k-k_1)^s} &= 2\left[\frac{k}{k-1}\right]^s + k^{s-1} \sum_{k_1=2}^{k-2} \frac{k}{k_1^s (k-k_1)^s} \\
&= 2\left[\frac{k}{k-1}\right]^s + k^{s-1} \left[\sum_{k_1=2}^{k-2} \frac{k-k_1}{k_1^s (k-k_1)^s} + \sum_{k_1=2}^{k-2} \frac{k_1}{k_1^s (k-k_1)^s}\right] \\
&= 2\left[\frac{k}{k-1}\right]^s + k^{s-1} \left[\sum_{k_1=2}^{k-2} \frac{1}{k_1^s (k-k_1)^{s-1}} + \sum_{k_1=2}^{k-2} \frac{1}{k_1^{s-1} (k-k_1)^s}\right] \\
&= 2\left[\frac{k}{k-1}\right]^s + 2 \sum_{k_1=2}^{k-2} \frac{k^{s-1}}{k_1^{s-1} (k-k_1)^s}.
\end{aligned}$$

Using the above we define

$$\phi_k^{(s)} := \sum_{k_1=2}^{k-2} \frac{k^{s-1}}{k_1^{s-1}(k-k_1)^s} = \frac{1}{2}\sum_{k_1=2}^{k-2} \frac{k^s}{k_1^s(k-k_1)^s}.$$

We then obtain the following recurrence inequality

$$\begin{aligned}
\phi_k^{(s)} &= \sum_{k_1=2}^{k-2} \frac{k^{s-1}}{k_1^{s-1}(k-k_1)^s} = k^{s-2}\sum_{k_1=2}^{k-2} \frac{(k-k_1)+k_1}{k_1^{s-1}(k-k_1)^s} \\
&= \frac{1}{k}\sum_{k_1=2}^{k-2} \frac{k^{s-1}}{k_1^{s-1}(k-k_1)^{s-1}} + \sum_{k_1=2}^{k-2} \frac{k^{s-2}}{k_1^{s-2}(k-k_1)^s} \\
&\leq \frac{1}{k}\sum_{k_1=2}^{k-2} \frac{k^{s-1}}{k_1^{s-1}(k-k_1)^{s-1}} + \frac{1}{2}\sum_{k_1=2}^{k-2} \frac{k^{s-2}}{k_1^{s-2}(k-k_1)^{s-1}} \quad = \quad \left[\frac{2}{k}+\frac{1}{2}\right]\phi_k^{(s-1)}.
\end{aligned}$$

Applying the above inequality $s-2$ times we get

$$\phi_k^{(s)} \leq \phi_k^{(2)}\left[\frac{2}{k}+\frac{1}{2}\right]^{s-2}.$$

Also

$$\begin{aligned}
\phi_k^{(2)} &= \sum_{k_1=2}^{k-2} \frac{k}{k_1(k-k_1)^2} = \sum_{k_1=2}^{k-2} \frac{1}{k_1(k-k_1)} + \sum_{k_1=2}^{k-2} \frac{1}{(k-k_1)^2} \\
&= \frac{1}{k}\left[\sum_{k_1=2}^{k-2}\frac{1}{k_1} + \sum_{k_1=2}^{k-2}\frac{1}{k-k_1}\right] + \sum_{k_1=2}^{k-2} \frac{1}{(k-k_1)^2} \\
&= \frac{2}{k}\sum_{k_1=2}^{k-2}\frac{1}{k_1} + \sum_{k_1=2}^{k-2}\frac{1}{k_1^2} \leq \frac{2}{k}\ln(k-2) + \frac{\pi^2}{6} - 1.
\end{aligned}$$

Using the above inequalities we get

$$\begin{aligned}
\sum_{k_1=1}^{k-1} \frac{k^s}{k_1^s(k-k_1)^2} &= 2\left[\frac{k}{k-1}\right]^s + 2\phi_k^{(s)} \leq 2\left[\frac{k}{k-1}\right]^s + 2\phi_k^{(2)}\left[\frac{2}{k}+\frac{1}{2}\right]^{s-2} \\
&\leq 2\left[\frac{k}{k-1}\right]^s + \left[\frac{4\ln(k-2)}{k} + \frac{\pi^2-6}{3}\right]\left[\frac{2}{k}+\frac{1}{2}\right]^{s-2} \quad = \quad \gamma_k. \quad \blacksquare
\end{aligned}$$

Lemma A.3. *Given $s \geq 2$ and $M \geq 6$, suppose there exist A_1, A_2 such that for every $j \in \{1,2\}$ and every $k \in \mathbb{Z}$, we have that $\left|c_k^{(j)}\right| \leq \frac{A_j}{\omega_k^s}$. Then, for any $k \in \mathbb{Z}$, we have that*

$$\left|\sum_{\substack{k_1+k_2=k \\ k_j\in\mathbb{Z}}} c_{k_1}^{(1)} c_{k_2}^{(2)}\right| \leq A_1 A_2 \frac{\alpha_k^{(2)}}{\omega_k^s}.$$

Proof. For $k = 0$ we have

$$\begin{aligned}
\sum_{\substack{k_1+k_2=0\\ k_j\in\mathbb{Z}}} \frac{1}{\omega^s_{k_1}\omega^s_{k_2}} &= \sum_{k_1=-\infty}^{-1} \frac{1}{\omega^s_{k_1}\omega^s_{-k_1}} + \frac{1}{\omega^s_0\omega^s_0} + \sum_{k_1=1}^{\infty} \frac{1}{\omega^s_{k_1}\omega^s_{-k_1}} \\
&= \frac{1}{\omega^s_0\omega^s_0} + 2\sum_{k_1=1}^{\infty} \frac{1}{\omega^s_{k_1}\omega^s_{k_1}} \\
&= 1 + 2\sum_{k_1=1}^{\infty} \frac{1}{k_1^{2s}} \le 4 + \frac{1}{2^{2s-1}(2s-1)}.
\end{aligned}$$

For $k > 0$ we have

$$\begin{aligned}
\sum_{\substack{k_1+k_2=k\\ k_j\in\mathbb{Z}}} \frac{1}{\omega^s_{k_1}\omega^s_{k_2}} &= \sum_{k_1=-\infty}^{-1} \frac{1}{\omega^s_{k_1}\omega^s_{k-k_1}} + \frac{2}{\omega^s_0\omega^s_k} + \sum_{k_1=1}^{k-1} \frac{1}{\omega^s_{k_1}\omega^s_{k-k_1}} + \sum_{k_1=k+1}^{\infty} \frac{1}{\omega^s_{k_1}\omega^s_{k-k_1}} \\
&= \sum_{k_1=1}^{\infty} \frac{1}{\omega^s_{k_1}\omega^s_{k+k_1}} + \frac{2}{\omega^s_0\omega^s_k} + \sum_{k_1=1}^{k-1} \frac{1}{\omega^s_{k_1}\omega^s_{k-k_1}} + \sum_{k_1=1}^{\infty} \frac{1}{\omega^s_{k_1}\omega^s_{k+k_1}} \\
&= \frac{1}{\omega^s_k}\left[2\sum_{k_1=1}^{\infty} \frac{\omega^s_k}{\omega^s_{k_1}\omega^s_{k+k_1}} + \frac{2}{\omega^s_0} + \sum_{k_1=1}^{k-1} \frac{\omega^s_k}{\omega^s_{k_1}\omega^s_{k-k_1}}\right] \\
&\le \frac{1}{\omega^s_k}\left[\frac{2}{\omega^s_0} + 2\sum_{k_1=1}^{\infty} \frac{1}{\omega^s_{k_1}} + \sum_{k_1=1}^{k-1} \frac{\omega^s_k}{\omega^s_{k_1}\omega^s_{k-k_1}}\right] \\
&= \frac{1}{k^s}\left[2 + 2\sum_{k_1=1}^{\infty} \frac{1}{k_1^s} + \sum_{k_1=1}^{k-1} \frac{k^s}{k_1^s(k-k_1)^s}\right] \\
&\le \frac{1}{k^s}\left[4 + \frac{2}{2^s} + \frac{2}{3^s} + \frac{2}{3^{s-1}(s-1)} + \sum_{k_1=1}^{k-1} \frac{k^s}{k_1^s(k-k_1)^s}\right].
\end{aligned}$$

In the two inequalities above we used integral estimates to bound the infinite sums. Using these inequalities and the upper bound γ_k from Lemma A.2 we have the result. ∎

Lemma A.4. *Given $s \ge 2$ and $M \ge 6$, suppose there exist $A_1, \dots, A_n$ such that for every $j \in \{1, \dots, n\}$ and every $k \in \mathbb{Z}$, we have that $\left|c^{(j)}_k\right| \le \frac{A_j}{\omega^s_k}$. Then, for any $k \in \mathbb{Z}$, we have that*

$$\left|\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\in\mathbb{Z}}} c^{(1)}_{k_1}\cdots c^{(n)}_{k_n}\right| \le \left(\prod_{j=1}^{n} A_j\right)\frac{\alpha^{(n)}_k}{\omega^s_k}.$$

Proof. For $k = 0$ we have

$$\begin{aligned}
\sum_{\substack{k_1+\cdots+k_n=0\\k_j\in\mathbb{Z}}} \frac{1}{\omega_{k_1}^s\cdots\omega_{k_n}^s} &= \sum_{k_1=-\infty}^{-1}\left[\frac{1}{\omega_{k_1}^s}\sum_{\substack{k_2+\cdots+k_n=-k_1\\k_j\in\mathbb{Z}}}\frac{1}{\omega_{k_2}^s\cdots\omega_{k_n}^s}\right]+\frac{1}{\omega_0^s}\sum_{\substack{k_2+\cdots+k_n=0\\k_j\in\mathbb{Z}}}\frac{1}{\omega_{k_2}^s\cdots\omega_{k_n}^s}\\
&+\sum_{k_1=1}^{\infty}\left[\frac{1}{\omega_{k_1}^s}\sum_{\substack{k_2+\cdots+k_n=-k_1\\k_j\in\mathbb{Z}}}\frac{1}{\omega_{k_2}^s\cdots\omega_{k_n}^s}\right]=\\
&\frac{1}{\omega_0^s}\sum_{\substack{k_2+\cdots+k_n=0\\k_j\in\mathbb{Z}}}\frac{1}{\omega_{k_2}^s\cdots\omega_{k_n}^s}+2\sum_{k_1=1}^{\infty}\left[\frac{1}{\omega_{k_1}^s}\sum_{\substack{k_2+\cdots+k_n=-k_1\\k_j\in\mathbb{Z}}}\frac{1}{\omega_{k_2}^s\cdots\omega_{k_n}^s}\right]\le\\
&\frac{1}{\omega_0^s}\frac{\alpha_0^{(n-1)}}{\omega_0^s}+2\sum_{k_1=1}^{\infty}\left[\frac{1}{\omega_{k_1}^s}\frac{\alpha_{k_1}^{(n-1)}}{\omega_{k_1}^s}\right]=\alpha_0^{(n-1)}+2\sum_{k_1=1}^{\infty}\frac{\alpha_{k_1}^{(n-1)}}{k_1^{2s}}\le\\
&\alpha_0^{(n-1)}+2\sum_{k_1=1}^{M-1}\frac{\alpha_{k_1}^{(n-1)}}{k_1^{2s}}+\frac{2\alpha_M^{(n-1)}}{(M-1)^{2s-1}(2s-1)}=\frac{\alpha_0^{(n)}}{\omega_0^s}.
\end{aligned}$$

For $k > 0$ we have

$$\begin{aligned}
\sum_{\substack{k_1+\cdots+k_n=k\\k_j\in\mathbb{Z}}} \frac{1}{\omega_{k_1}^s\cdots\omega_{k_n}^s} &= \sum_{k_1=-\infty}^{-1}\left[\frac{1}{\omega_{k_1}^s}\sum_{\substack{k_2+\cdots+k_n=k-k_1\\k_j\in\mathbb{Z}}}\frac{1}{\omega_{k_2}^s\cdots\omega_{k_n}^s}\right]+\frac{1}{\omega_0^s}\sum_{\substack{k_2+\cdots+k_n=k\\k_j\in\mathbb{Z}}}\frac{1}{\omega_{k_2}^s\cdots\omega_{k_n}^s}\\
&+\sum_{k_1=1}^{k-1}\left[\frac{1}{\omega_{k_1}^s}\sum_{\substack{k_2+\cdots+k_n=k-k_1\\k_j\in\mathbb{Z}}}\frac{1}{\omega_{k_2}^s\cdots\omega_{k_n}^s}\right]+\frac{1}{\omega_k^s}\sum_{\substack{k_2+\cdots+k_n=0\\k_j\in\mathbb{Z}}}\frac{1}{\omega_{k_2}^s\cdots\omega_{k_n}^s}\\
&+\sum_{k_1=k+1}^{\infty}\left[\frac{1}{\omega_{k_1}^s}\sum_{\substack{k_2+\cdots+k_n=k-k_1\\k_j\in\mathbb{Z}}}\frac{1}{\omega_{k_2}^s\cdots\omega_{k_n}^s}\right]\\
&\le \sum_{k_1=1}^{\infty}\left[\frac{1}{\omega_{k_1}^s}\frac{\alpha_{k+k_1}^{(n-1)}}{\omega_{k+k_1}^s}\right]+\sum_{k_1=1}^{k-1}\left[\frac{1}{\omega_{k_1}^s}\frac{\alpha_{k-k_1}^{(n-1)}}{\omega_{k-k_1}^s}\right]+\sum_{k_1=1}^{\infty}\left[\frac{1}{\omega_{k+k_1}^s}\frac{\alpha_{k_1}^{(n-1)}}{\omega_{k_1}^s}\right]\\
&+\frac{1}{\omega_0^s}\frac{\alpha_k^{(n-1)}}{\omega_k^s}+\frac{1}{\omega_k^s}\frac{\alpha_0^{(n-1)}}{\omega_0^s}.
\end{aligned}$$

Consider $k \in \{1, \dots, M-1\}$. Since $\alpha_{k_1}^{(n-1)} \le \alpha_M^{(n-1)}$, for all $k_1 \ge M$, we have

$$\begin{aligned}
\sum_{k_1=1}^{\infty}\frac{\alpha_{k+k_1}^{(n-1)}}{\omega_{k_1}^s\omega_{k+k_1}^s} &= \sum_{k_1=1}^{M-k-1}\frac{\alpha_{k+k_1}^{(n-1)}}{\omega_{k_1}^s\omega_{k+k_1}^s}+\sum_{k_1=M-k}^{\infty}\frac{\alpha_{k+k_1}^{(n-1)}}{\omega_{k_1}^s\omega_{k+k_1}^s}\\
&\le \sum_{k_1=1}^{M-k-1}\frac{\alpha_{k+k_1}^{(n-1)}}{\omega_{k_1}^s\omega_{k+k_1}^s}+\alpha_M^{(n-1)}\sum_{k_1=M-k}^{\infty}\frac{1}{\omega_{k_1}^s\omega_{k+k_1}^s}\\
&\le \sum_{k_1=1}^{M-k-1}\frac{\alpha_{k+k_1}^{(n-1)}}{\omega_{k_1}^s\omega_{k+k_1}^s}+\alpha_M^{(n-1)}\sum_{k_1=1}^{\infty}\frac{1}{k_1^s(k+k_1)^s}\\
&\le \frac{1}{k^s}\left[\sum_{k_1=1}^{M-k-1}\frac{\alpha_{k+k_1}^{(n-1)}k^s}{\omega_{k_1}^s\omega_{k+k_1}^s}+\alpha_M^{(n-1)}\left(\frac{1}{(M-k)^sM^s}+\frac{1}{(M-k)^{s-1}(M+1)^s(s-1)}\right)\right].
\end{aligned}$$

Similarly,

$$\sum_{k_1=1}^{\infty} \frac{\alpha_{k_1}^{(n-1)}}{\omega_{k_1}^s \omega_{k+k_1}^s} \le \frac{1}{k^s} \left[\sum_{k_1=1}^{M-1} \frac{\alpha_{k_1}^{(n-1)} k^s}{\omega_{k_1}^s \omega_{k+k_1}^s} + \frac{\alpha_M^{(n-1)}}{(M-1)^{s-1}(s-1)} \right].$$

From the definition of $\alpha_k^{(n)}$ for $k \in \{1, \ldots, M-1\}$, it follows that

$$\sum_{\substack{k_1+\cdots+k_n=k \\ k_j \in \mathbb{Z}}} \frac{1}{\omega_{k_1}^s \cdots \omega_{k_n}^s} \le \frac{\alpha_k^{(n)}}{\omega_k^s}.$$

Consider now $k \ge M$, then

$$\sum_{k_1=1}^{\infty} \frac{\alpha_{k+k_1}^{(n-1)}}{\omega_{k_1}^s \omega_{k+k_1}^s} + \frac{\alpha_k^{(n-1)}}{k^s} \le \frac{\alpha_M^{(n-1)}}{k^s} \left[2 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{3^{s-1}(s-1)} \right].$$

Using Lemma A.2, we get that

$$\begin{aligned}
\sum_{k_1=1}^{k-1} \frac{\alpha_{k_1}^{(n-1)}}{\omega_{k_1}^s \omega_{k-k_1}^s} &= \sum_{k_1=1}^{M-1} \frac{\alpha_{k_1}^{(n-1)}}{\omega_{k_1}^s \omega_{k-k_1}^s} + \frac{1}{k^s} \sum_{k_1=M}^{k-1} \frac{k^s \alpha_{k_1}^{(n-1)}}{\omega_{k_1}^s \omega_{k-k_1}^s} \\
&\le \frac{1}{k^s} \sum_{k_1=1}^{M-1} \frac{\alpha_{k_1}^{(n-1)}}{\omega_{k_1}^s \left(1 - \frac{k_1}{k}\right)^s} + \frac{\alpha_M^{(n-1)}}{k^s} \sum_{k_1=M}^{k-1} \frac{k^s}{\omega_{k_1}^s \omega_{k-k_1}^s} \\
&\le \frac{1}{k^s} \left[\sum_{k_1=1}^{M-1} \frac{\alpha_{k_1}^{(n-1)}}{\omega_{k_1}^s \left(1 - \frac{k_1}{M}\right)^s} + \alpha_M^{(n-1)} \gamma_k \right].
\end{aligned}$$

Also,

$$\sum_{k_1=1}^{\infty} \frac{\alpha_{k_1}^{(n-1)}}{\omega_{k_1}^s \omega_{k+k_1}^s} \le \frac{1}{k^s} \left[\sum_{k_1=1}^{M-1} \frac{\alpha_{k_1}^{(n-1)}}{\omega_{k_1}^s} + \frac{\alpha_M^{(n-1)}}{(M-1)^{s-1}(s-1)} \right].$$

Combining the above inequalities, we get

$$\begin{aligned}
&\sum_{\substack{k_1+\cdots+k_n=k \\ k_j \in \mathbb{Z}}} \frac{1}{\omega_{k_1}^s \cdots \omega_{k_n}^s} \\
&\quad \le \frac{1}{k^s} \Bigg[\alpha_0^{(n-1)} + \sum_{k_1=1}^{M-1} \frac{\alpha_{k_1}^{(n-1)}}{k_1^s} \left(1 + \frac{1}{\left(1 - \frac{k_1}{M}\right)^s} \right) \\
&\qquad\qquad + \alpha_M^{(n-1)} \left(2 + \tfrac{1}{2^s} + \tfrac{1}{3^s} + \tfrac{1}{3^{s-1}(s-1)} + \tfrac{1}{(M-1)^{s-1}(s-1)} + \gamma_k \right) \Bigg] = \frac{\alpha_k^{(n)}}{\omega_k^s}. \quad \blacksquare
\end{aligned}$$

Remark A.5. *Note that the $\alpha_k^{(n)}$ provides, for $1 \le |k| \le M-1$, a slight improvement over the $\alpha_k^{(p)}$ defined in Section* **A.2** *in [9]. The difference comes from the following upper bound*

$$\sum_{k_1=M-k}^{\infty} \frac{1}{k_1^s (k+k_1)^s} \le \frac{1}{(M-k)^s M^s} + \frac{1}{(M-k)^{s-1}(M+1)^s(s-1)}.$$

In [9], the coarser upper bound

$$\sum_{k_1=M-k}^{\infty} \frac{1}{k_1^s (k+k_1)^s} \le \frac{1}{k^s} \left[1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{3^{s-1}(s-1)} \right]$$

is used to construct $\alpha_k^{(p)}$.

The following corollary of Lemma A.4 gives better bounds for the cases $0 \le |k| \le M-1$. Given $s \ge 2$ and $M \ge 6$ we define, for $k \ge 0$,

$$\varepsilon_k^{(n)} = \varepsilon_k^{(n)}(s,M) := \frac{2\alpha_M^{(n-1)}}{(s-1)(M-1)^{s-1}(M+k)^s} + \sum_{k_1=M}^{M+k-1} \frac{\alpha_{k_1-k}^{(n-1)}}{\omega_{k_1}^s \omega_{k_1-k}^s} \tag{42}$$

and for $k < 0$

$$\varepsilon_k^{(n)}(s,M) := \varepsilon_{|k|}^{(n)}(s,M).$$

Corollary A.6. *Given $s \ge 2$ and $M \ge 6$, for $n \ge 3$ and $0 \le |k| \le M-1$ we have that*

$$\left| \sum_{\substack{k_1+\cdots+k_n=k \\ k_j \in \mathbb{Z}}} c_{k_1}^{(1)} \cdots c_{k_n}^{(n)} \right| \le \left| \sum_{\substack{k_1+\cdots+k_n=k \\ |k_j|<M}} c_{k_1}^{(1)} \cdots c_{k_n}^{(n)} \right| + n \left(\prod_{i=1}^n A_i \right) \varepsilon_k^{(n)}.$$

Proof. Notice that

$$\sum_{\substack{k_1+\cdots+k_n=k \\ k_j \in \mathbb{Z}}} c_{k_1}^{(1)} \cdots c_{k_n}^{(p)} = \sum_{\substack{k_1+\cdots+k_n=k \\ |k_j|<M}} c_{k_1}^{(1)} \cdots c_{k_n}^{(n)} + \sum_{\substack{k_1+\cdots+k_n=k \\ \max\{|k_j|\} \ge M}} c_{k_1}^{(1)} \cdots c_{k_n}^{(n)}.$$

Without loss of generality, suppose that $|k_1| \ge M$ in the second sum. Then

$$\begin{aligned}
\sum_{\substack{k_1+\cdots+k_n=k \\ |k_1| \ge M}} \frac{1}{\omega_{k_1}^s \cdots \omega_{k_n}^s} &= \sum_{k_1=-\infty}^{-M} \frac{1}{\omega_{k_1}^s} \sum_{k_2+\cdots+k_n=k-k_1} \frac{1}{\omega_{k_2}^s \cdots \omega_{k_n}^s} \\
&\qquad + \sum_{k_1=M}^{\infty} \frac{1}{\omega_{k_1}^s} \sum_{k_2+\cdots+k_n=k-k_1} \frac{1}{\omega_{k_2}^s \cdots \omega_{k_n}^s} \\
&\le \sum_{k_1=M}^{\infty} \left[\frac{\alpha_{k+k_1}^{(n-1)}}{\omega_{k_1}^s \omega_{k+k_1}^s} + \frac{\alpha_{k_1-k}^{(n-1)}}{\omega_{k_1}^s \omega_{k_1-k}^s} \right] \\
&\le \left[2\alpha_M^{(n-1)} \sum_{k_1=M}^{\infty} \frac{1}{\omega_{k_1}^s \omega_{k+k_1}^s} + \sum_{k_1=M}^{M+k-1} \frac{\alpha_{k_1-k}^{(n-1)}}{\omega_{k_1}^s \omega_{k_1-k}^s} \right] \\
&\le \left[\frac{2\alpha_M^{(n-1)}}{(M+k)^s (M-1)^{s-1}(s-1)} + \sum_{k_1=M}^{M+k-1} \frac{\alpha_{k_1-k}^{(n-1)}}{\omega_{k_1}^s \omega_{k_1-k}^s} \right].
\end{aligned}$$

The result then follow from the definition of $\varepsilon_k^{(n)}$. ∎

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