Theorem. *Let $q = X^T AX$ be a quadratic form where A is an $n \times n$ symmetric matrix, and let λ_1 be the least eigenvalue of A and λ_2 the greatest eigenvalue of A. Then*

1. $\min\{q(X) \mid \|X\| = 1\} = \lambda_1$, *and* $q(v_1) = \lambda_1$ *where* v_1 *is any unit eigenvector corresponding to* λ_1.

2. $\max\{q(X) \mid \|X\| = 1\} = \lambda_2$, *and* $q(v_2) = \lambda_2$ *where* v_2 *is any unit eigenvector corresponding to* λ_2.

One can read the proof in Nicholson, Theorem 5 of 4.8.3. Here is an example of how to use this:

Example. Let $q(x_1, x_2, x_3) = 5x_1^2 + 8x_2^2 + 5x_3^2 - 4x_1x_2 + 8x_2x_3 + 4x_2x_3$. Find the maximum and minimum values of q on the unit sphere (i.e. where X is restricted to having $\|X\| = 1$), and a point where each value is achieved.

Solution. Write q in matrix form as $q = X^T AX$ where $A = \begin{pmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{pmatrix}$.

We find that A has eigenvalues $\lambda_1 = 0, \lambda_2 = \lambda_3 = 9$, and

$$\begin{aligned} E_{\lambda_1} &= \text{span}\{v_1 = [2\ 1\ -2]^T\} \\ E_{\lambda_2} &= \text{span}\{v_2 = [1\ -2\ 0]^T, v_3 = [1\ 0\ 1]^T\} \end{aligned}$$

Hence the minimum of q on the unit sphere is 0 (the least eigenvalue) and is achieved at $p_1 = \frac{v_1}{\|v_1\|} = \frac{1}{3}[2\ 1\ -2]^T$ (also at $-p_1$). The maximum is 9 (the greatest eigenvalue) and is achieved at any linear combination of v_2 and v_3 that is a unit vector, for example $p_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{5}}[1\ -2\ -0]^T$.