Last Name (PRINT) : ____________________ Student Number : __________

Other name(s) : ____________________

FACULTY OF SCIENCE – FINAL EXAMINATION

MATHEMATICS 133 – VECTORS, MATRICES AND GEOMETRY

VERSION 1

Examiner: Professor I. Klemes
Associate Examiner: Professor W. Jonsson

Date: Friday, 18 April 2008.
Time: 9:00 A.M. - 12:00 noon.

INSTRUCTIONS

1. Check that this paper has a total of 12 pages, which includes this cover and the three continuation pages at the end. You must not tear out any pages.
2. This is a closed book exam. Notes, dictionaries of any kind, calculators, or any other devices, are not permitted.
3. Information on the computer card is to be entered with a soft lead pencil. Any erasing must be done cleanly. The computer will also accept black or blue ball point pens, but this is not advised as you will not be able to erase these to make corrections.
4. This exam paper is **Version 1**. Make sure that the "Version column" of your computer card has this same number filled in.
5. Print your name and student number in the blanks at the top of this exam paper.
6. Enter the requested ID information on the computer card, including the "check bits" columns as instructed on the card. Sign the computer card in the space indicated.
7. This exam paper and the computer card may **not** be removed from the exam room and must be handed in.
8. This exam paper has a total of 100 points and consists of two parts:
   Part I consists of 12 multiple choice questions worth 40/12 points each, for a total of 40. **Only the answers entered on the computer card will count for Part I.**
   Part II consists of 5 written questions together worth 60 points. In this part, show all of the work necessary for each step in the solution, and simplify answers.
9. Your solutions to the questions of Part II are to be written in the blank spaces provided in this exam paper below each question. *You may do your rough work on the facing page (i.e. back of the previous sheet), or you may use it to continue your work.* Further work may be continued on the two blank continuation pages at the end of this exam paper. You must clearly indicate where your work continues.
10. The examination Security Monitor Program detects pairs of students with unusually similar answer patterns on multiple choice exams. Data generated by this program can be used as admissible evidence, either to initiate or corroborate an investigation or a charge of cheating under Section 16 of the Code of Student Conduct and Disciplinary Procedures.

Please do not write inside the box :

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Part I: Answer on the computer card. Only the answer on the computer card will be considered. For each question the correct answer is assigned the same weight (40/12 points). Anything else (wrong answer, more than one answer, no answer, answer not readable by the machine, etc.) is assigned zero.

1. Let A be an 8×5 matrix such that the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$. What is the dimension of the row space of A ?

   (A) 0 (B) 3 (C) 5 (D) 7 (E) 8

2. Let $A = \begin{bmatrix} 3 & s \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & t \\ 0 & -2 \end{bmatrix}$ where s and t are real numbers.

   Which one of the following statements is true ?

   (A) A is diagonalizable for all s, and B is diagonalizable only if $t = 0$.

   (B) A is diagonalizable if $s = 0$, and B is not diagonalizable if $t \neq 0$.

   (C) A is not diagonalizable if $s \neq 0$, and B is diagonalizable for all t.

   (D) A and B are both diagonalizable for all s and all t.

   (E) A and B are diagonalizable only if $s = 0$ and $t = 0$.

3. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$ such that $||\mathbf{u}|| = 2$, $||\mathbf{v}|| = 5$, and $\mathbf{u} \cdot \mathbf{v} = 3$. Then $||\mathbf{u} - \mathbf{v}|| =$

   (A) $\sqrt{35}$ (B) $\sqrt{43}$ (C) 43 (D) $\sqrt{23}$ (E) 29

4. An equation of the plane containing the point $P(1, -1, 2)$ and perpendicular to the line $(x, y, z) = (2 - 3t,\ 1 + 5t,\ 7 + t)$ is

   (A) $2x + y + 7z = 15$ (B) $6x - 10y - 2z = 12$ (C) $4x - 9y - 6z = 1$

   (D) $9x + 5y - 2z = 0$ (E) $3x + 2y - z = -1$

5. If $\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 12$ then $\det \begin{bmatrix} 2a_{11} - a_{31} & 2a_{12} - a_{32} & 2a_{13} - a_{33} \\ a_{21} + 3a_{11} & a_{22} + 3a_{12} & a_{23} + 3a_{13} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{bmatrix}$ has the value

   (A) 48 (B) -48 (C) 0 (D) 144 (E) -148

6. The plane π has the vector equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$. A vector perpendicular to this plane is

   (A) $[1, -1, 2]^T$ (B) $[1, -1, 1]^T$ (C) $[7, 9, 1]^T$ (D) $[1, 0, -2]^T$ (E) $[-1, 7, -9]^T$

7. If A is a 3×3 matrix such that $\det(A) = 2$, find $\det(5AA^T)$.

   (A) 5 (B) 1000 (C) 100 (D) 20 (E) 500

8. For a 3×3 matrix A, suppose that $A^3 = -I$ where I is the 3×3 identity matrix. What is $(A - I)^{-1}$?

   (A) $A^2 - A + I$ (B) $A - I$ is not invertible

   (C) $-\frac{1}{2}(A^2 + A + I)$ (D) $\frac{1}{2}(A^2 - A + I)$ (E) $-\frac{1}{2}I$

9. If $S : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection onto the line $6x + 2y = 0$, which of the following is a pair of eigenvectors of S ?

   (A) $\begin{bmatrix} 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -6 \end{bmatrix}$ (B) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ (C) $\begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

   (D) $\begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 9 \end{bmatrix}$ (E) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

10. The vectors $\begin{bmatrix} -2 \\ -5 \\ -3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \\ 2 \\ 5+k \end{bmatrix}$, $\begin{bmatrix} -2 \\ 4 \\ 6 \\ -8 \end{bmatrix}$ are linearly independent if and only if

(A) $k \neq -7$, (B) $k \neq 0$, (C) $k = 0$, (D) $k \neq -4$, (E) $k = -5$

11. Let $A = \begin{bmatrix} -1 & a \\ a & a-2 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then the linear system $AX = B$ has:

(A) a unique solution if and only if $a \neq -2$.
(B) a unique solution if and only if $a \neq 1$.
(C) more than one solution if $a = -2$.
(D) no solution if $a = 2$.
(E) more than one solution if $a = 1$.

(As usual, only one of the above statements is true!)

12. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$.

Which one of the following statements is true ?

(A) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbb{R}^3$,

(B) $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of Span $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -6 \\ 0 \end{bmatrix} \right\}$.

(C) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis of Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$,

(D) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$,

(E) $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of Span$\{\mathbf{v}_3, \mathbf{v}_4\}$.

Part II: Answer in the space provided below each question. Continue on the facing page (i.e. back of the previous page). After that, use one of the continuation pages at the end of this booklet, being sure to indicate its location.

II.1 (10 points)

(a) Find the distance from the point $P(2,1,0)$ to the line L determined by the two points $A(1,1,1)$ and $B(2,-1,3)$.

(b) Find the point Q on the line L which is closest to the point P in part (a).

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then on page number: ______.

Exam continues overleaf...

II.2 (15 points) Consider the matrix

$$M = \begin{bmatrix} 1 & 1 & 4 & -3 & 7 \\ 1 & 2 & 5 & -5 & 8 \\ 1 & 2 & 5 & -6 & 11 \\ 3 & 5 & 14 & -13 & 23 \\ 1 & 3 & 6 & -7 & 9 \end{bmatrix}.$$

(a) Find the **reduced** echelon form of M, showing your work. (You should find that M has rank 3.)

(b) Find a basis for the column space of M.

(c) Find a basis for the row space of M.

(d) Find a basis for the null space of M.

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then on page number: ______.

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II.3 (10 points)

(a) Find the standard matrix of the linear transformation $R : \mathbb{R}^2 \to \mathbb{R}^2$ defined by reflection in the line $2x + 5y = 0$.

(b) Suppose that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ has eigenvalues $\lambda = 2$ and $\lambda = 3$, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively. Find the standard matrix of T.

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II.4 (15 points)

Let $A = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 3 & -2 \\ 1 & 0 & 1 \end{bmatrix}$.

(a) Find the characteristic polynomial (its factored form will be sufficient), and find all eigenvalues of A.

(b) Show that one of the eigenspaces of A has dimension 2.

(c) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$
(do not compute P^{-1}).

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II.5 (10 points)

Consider the quadratic form $q(X) = X^T AX$ where $A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(a) Find an orthogonal matrix P such that the change of variables $X = PY$ converts q to the diagonal form $q(X) = ay_1^2 + by_2^2$.

(b) Find the minimum and maximum values achieved by $q(X)$ on the unit circle $x_1^2 + x_2^2 = 1$.

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Continuation of solution for problem: ______.
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