LECTURE 17

Theorem 1. *Let G be finitely presented $C(4)-T(4)$ group. Then G is automatic.*

PROOF. We prove special case when $G = \pi_1 X$ with X a $(4,4)$ complex. Without loss of generality, each 2-cell is a square (obtain this by subdividing). Without loss of generality, X has one 0-cell (treat general case using "$A * B$ is automatic $\Leftrightarrow A, B$ are automatic" and $\pi_1(X/X^0) \cong \pi_1 X * f_{|X^0|-1}$).

The regular language L consists of geodesic paths in $(\widetilde{X})'$ which do not contain certain subpaths. ($\widetilde{X}$ = Cayley graph of G.)

Forbidden paths:

> (figure) no "backtracks" (in fact, already forbidden by geodesic requirement),
> (figure) no "wrap around" a single 2-cell (in fact, already forbidden by geodesic requirement),
> (figure) no "postponed turns."

We need to show that for each $v \in \tilde{X}^0$ there is a nonforbidden geodesic path from 1 to v. Show that starting with a geodesic γ.

(figure) Gradually transforming the geodesic obtain nonforbidden one. Process stops because each transformation yields a geodesic less with respect to lexicographical order, induced by assigning "t" to turn vertices and "n" to non-turn vertices.

Obtained a language L that surjects to G. Interestingly, it is language we were talking about last time. (figure)

Prove fellow travelling property. Suppose U, V are paths in L such that $d_G(u, v) \leq 1$.

(figure)

Then there is a disc diagram for uev^{-1} where $e = 1$ or a single edge.

Combinatorial Gauss–Bonet theorem: curvature argument implies D is a ladder, so U, V fellow travel. Namely, note that curvature given by the top path is at most $\frac{\pi}{2}$, as is curvature given by the bottom. Indeed,

> path can't have $\frac{\pi}{2}$ curvature right after 0 (figure),
> path can't have two $\frac{\pi}{2}$ in a row (figure),
> can't happen (figure),
> after string of 0, there is a negative curvature before positive (figure).

(figure) Sum is 2π, so at the "end" we have two positive angles, which gives a ladder.

Note that π curvature at the "start" may be ignored, as language is closed under subpath (figure), and note also that $\frac{\pi}{2}$ curvature at the "start" gives nonpositive curvature on the top and bottom (figure).

□

Definition 1. *G is biautomatic if there is a language in monoid generators S of G such that*

> *1) $L \to G$ is a surjection,*
> *2) K-fellow travelling property between the paths U, V such that initial and end points of U, V are at distance ≤ 1 in $\Gamma(G, S)$.*

Definition 2. *A combing of a group G with monoid generators S is a map $\phi : G \to M$ $(M = M(S))$ such that with $\rho : M \to G$ induced by generators, $\rho \circ \phi = \mathrm{Id}$ (i.e. it's assignment of normal form for each $g \in G$). (figure)*

Require that:
1) $|\phi(g)| \leq K d(1, g)$ for some universal constant,
2) if $d(g, h) \leq 1$, then $\phi(g), \phi(h)$ K-fellow travel.
1') $\phi(g)$ is quasigeodesic.

For bicombing, require
2') left + right fellow travel.

CAT(0) groups are combable, automatic groups are combable. Unknown if a combable group is always automatic.

If G is combable then

1) G is finitely presented,
2) G has a quadratic isoperimetric function.

Draw all combings, use fellow travelling.

(figure)