Automatic Groups.

Definition 1. *Finite State Automata is a finite directed graph* Γ*, labeled by some alphabet* $S \subset \{a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}\}$*. Vertices are called states. One vertex is a start state. Some vertices are accept states.*

EXAMPLE. (figure)

Language L accepted by the FSA is set of all words in $M = M(S)$ (a free monoid) that can be "read off" from Γ starting at a start state, ending at an accept state.

So accepted words correspond to labels of directed paths starting at start, ending at accept state.

The FSA Γ is deterministic if there is at most one outgoing edge labeled s at each state.

Languages L accepted by a FSA are called regular.

(figure) $L = \{a^n | n \geq 0, n =_3 2\}$.

$S = \{0, 1\}$.

Any finite subset is regular. (figure)

Lemma 1. *Any regular language is actually given by a deterministic FSA.*

PROOF. Make a new FSA whose states are collections of states in Γ. (Endpoint of arrow starting at a collection is collection of corresponding endpoints.)

Lemma 2. *If* L *is regular then* L^C *(compliment) is also regular.*

PROOF. Let Γ be DFSA such that $L = \text{Accept}(\Gamma)$. Let Γ^C be FSA on the same labeled directed graph as Γ with the same start state, but with accept states concluding compliment set to set of accept states Γ. Then language accepted by Γ^C is L^C.

(figure: example of Γ^C)

Consider Cayley graph J of group $\mathbb{Z} \times \mathbb{Z}$.

(figure: Cayley graph) Finite geodesics are given by the rule "never go back". Set of all finite geodesics is regular language.

(figure: corresponding FSA)

Theorem 1. *(A Sample Theorem) Suppose* $L = L(S)$ *is a regular language in some alphabet* S*. Let* $\phi : S \to G$ *be a map to finitely presented group. Let* $\phi : L \to G$ *be the induced map. Suppose* $\phi : L \to G$ *is surjective and finite-to-one, i.e.* $|f^{-1}(g)| < \infty$*. Then* G *has solvable word problem.*

PROOF. Given a word w in generators $\phi(s_1), \ldots, \phi(s_k)$ of G we can recognize $w = 1_G$ by enumerating possibilities.

How to recognize $w \neq 1_G$? There are finitely many words $\{u_1, \ldots u_k\}$ such that $\phi(u_i) = 1$. By brute searching we eventually determine that $w = \phi(l)$ for some word $l \in L$ (since we can enumerate all words in L). Then check if $l \in \{u_1, \ldots, u_k\}$.

Automatic Structure for a Group G.

Definition 2. *Automatic structure for a group* G, S *(*S *is set of monoid generators) a regular language* L *in* $S = \{s_1, \ldots, s_k\}$ *such that:*

1) *the map* $L \to G$ *is surjective,*
2) $\exists K$ *such that if* $u, v \in L$ *and* $d_g(u, v) \leq 1$ *then the paths* u, v *in* $\Gamma(G, S)$ K*-fellow travel.*

(figure: fellow traveling) $d(u_i, v_i) < k$
EXAMPLE. $\mathbb{Z} \times \mathbb{Z}$. $L = \{a^n b^m | n, m \in \mathbb{Z}\}$
(figure)
EXAMPLE. "Diagonal, then straight". $(ab)^n a^m, (ab)^n b^m, (ba)^n a^m, (ba)^n b^m$, etc.
(figure)

Theorem 2. *(ECHLPT) Let G be word-hyperbolic with monoid generating set S.*
1) Then set of all geodesics in S form a regular language.
2) L is the regular language of an automatic structure for (G, S).

PROOF. 1) Holds since σ not a geodesic $\Leftrightarrow \exists w_i \subset \sigma$ such that $w_i \in \{w_1, \ldots, w_k\}$ $(w_1, \ldots, w_k$ are forbidden set, i.e. larger part of relators).

EXERCISE. Set of words not containing any $\{w_1, \ldots, w_k\}$ is regular.

2) Thin triangles. (figure) $\square$

In particular, this applies to finite groups.

FACT. If A, B are automatic, then $A \times B$ is automatic. But it is unknown whether $A \times B$ automatic implies A, B automatic.

Theorem 3. *Finitely presented $C(4) - T(4), C(6) - T(3), C(3) - T(6)$ groups are automatic.*

Theorem 4. *If G is automatic, then its isoperimetric function is $\leq n^2$.*

So in some sense it's "nonpositive" curvature.