Lecture 15

Rips's Short Exact Sequence.

$$1 \to N \to G \to Q \to 1$$

$$1 \to \langle x, y \rangle \to \langle x, y, g_1, \ldots, g_s | R_1, \ldots, R_t \rangle \to \langle \bar{g}_1, \ldots, \bar{g}_s | \overline{R}_1, \ldots, \overline{R}_t \rangle \to 1$$

Note. N is finitely presented $\Leftrightarrow Q$ is finite.

Example. Finitely generated normal subgroup of a free group F_n is either trivial or finite index. (figure)

Theorem 1. *(Bieri, Strebel) If X is a finite aspherical 2-complex and $N \lhd \pi_1 X$ with N finitely presented then either*

0) N is trivial
1) N is free and $1 \to N \to \pi_1 X \to \binom{\text{virtually}}{\text{free}} \to 1$
2)N has finite index in $\pi_1 X$

Example. There is a map $Q \to \mathrm{Out}(N) = \mathrm{Aut}(N)/\mathrm{In}(N)$

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Aut}(N) \\ \downarrow & & \downarrow \\ G/N & \longrightarrow & \mathrm{Aut}(N)/\mathrm{In}(N) \end{array}$$

The map $Q \to \mathrm{Out}(N)$ is injective in our case. With a bit of a extra work (modify construction) arrange $Q \simeq \mathrm{Out}(N)$. So every finitely presented group $\cong \mathrm{Out}(N)$ for some finitely generated N.

Example. $N = G \Leftrightarrow Q$ is trivial.

Fact. There is no algorithm to detect if a group determined by finite presentation is nontrivial. Thus "generation problem" for G is undecidable (in general).

$F_2 \star F_2$ has undecidable generation problem, membership problem.

The Construction. Consider presentation

$$\left\langle \begin{matrix} q_1, \ldots, q_s \\ x, y \end{matrix} \middle| \begin{matrix} x_{q_i} = X_{i+} & x_{q_i^{-1}} = X_{i-} \\ y_{q_i} = Y_{i+} & y_{q_i^{-1}} = Y_{i-} \end{matrix} \right\rangle,$$

where $X_{i+}, X_{i-}, Y_{i+}, Y_{i-}$ are words in x, y. If $X_{i\pm}, Y_{i\pm}$ are chosen randomly of length < 1000000 then we obtain $C'(\frac{1}{6})$ presentation, and subgroup $\langle x, y \rangle$ is normal but nontrivial. ($C'(\frac{1}{6}) \Rightarrow$ It's Dehn's presentation $\Rightarrow x$ is nontrivial, since too short.)

$$\left\langle \begin{matrix} q_1, \ldots, q_s \\ x, y \end{matrix} \middle| \begin{matrix} \left. \begin{matrix} x_{q_i} = X_{i+} & x_{q_i^{-1}} = X_{i-} \\ y_{q_i} = Y_{i+} & y_{q_i^{-1}} = Y_{i-} \end{matrix} \right\} \\ R_i = W_i \end{matrix} \quad \begin{matrix} \text{make } N \\ \text{normal} \\ \text{make } G/N \cong Q \end{matrix} \right\rangle,$$

where W_i are words in x, y, with lengths $> 10(\max\{R_i\})$.

For example,

$$\begin{array}{rcl} X_{i+} & = & xyxy^2 \cdots xy^{1000M} \\ X_{i-} & = & xy^{1000M+1}x \cdots xy^{2000M} \\ Y_{i+} & = & xy^{2000M+1}x \cdots xy^{3000M} \\ & & \ldots \end{array}$$

this way pieces are tiny.

Fact. (Guba) Let X and Y be reduced cyclically reduced cyclically distinct words in $a_i^{\pm 1}$ which are not proper powers. Suppose U has a subword both of X^n and Y^n. Then $|U| \leq |X| + |Y|$.

Theorem 2. *Let $\langle a_1, \dots, a_n | R_1, \dots, R_m \rangle$ (R_i are not proper powers) be a presentation with $R_i \not\sim R_j^{\pm 1}$. Then $\langle a_1, \dots, a_n | R_1^{d_1}, \dots, R_m^{d_m} \rangle$ is $C'(\frac{1}{6})$ provided $|R_i^{d_i}| > 6(max_j\{|R_i|\})$ for each i.*

The Conjugacy Problem. Given $\langle a_1, \dots a_s | R_1, \dots, R_t \rangle$, words U, V in $a_i^{\pm 1}$. Is $U \sim_G V$. This problem is (in general) hard, is not a geometric problem. In particular, exists G with finite index subguoup H such that $CP(G)$ solvable, $CP(H)$ not ($CP(H)$ solvable, $CP(G)$ not). This is opposite to word problem, for which G_1 quasiisometric to G_2 implies $WP(G_1)$ decidable $\Leftrightarrow WP(G_2)$ decidable.

Geometric Interpretation. Annular diagram A is a finite 2-complex with specific planar embedding such that $\pi_1 A \cong \mathbb{Z}$.

(figure: examples)

Annular diagram has 2 boundary paths.

(figure)

Annular diagram in X is just a combinatorial map $A \to X$.

$A \to X$ is reduced if it contains no cancellable pairs. (Obtained by removing cancelable pairs.) Area (A) is number of its 2-cells.

Theorem 3. *Paths P_1 and P_2 are homotopic to each other (P_1 and P_2 are conjugated in $\pi_1 X$) iff exists an annular diagram in X with boundary paths P_1, P_2.*

Proof. Suppose A exists, then $P_1 \sim P_2$. Indeed, $P_1 \sim P_2$ in A, this projects to a homotopy in X.

Conversely, if $P_1 \sim P_2$ in X then we construct A by e.g. $P_1 = QP_2Q^{-1}$ for some path Q, so exists disc diagram $D \to X$ with $\partial_p D = P_1 Q P_2^{-1} Q^{-1}$. Now form A by identifying $Q = Q$.

(figure: identifying)

When X is small-cancellation then a minimal area diagram is also small cancellation. But $C'(\frac{1}{6})$ diagrams are very limited because of Combinatorial Gauss–Bonet Theorem.

Greendlinger's Lemma.

Theorem 4. *Any $C(6)$ annular diagram must contain an i-shell or spur with $i \leq 4$. Any $C(4) - T(4)$ annular diagram must contain an i-shell or spur with $i \leq 3$.*

Proof. (Similar argument to Greendlinger's Lemma.)

$$\begin{aligned} 2\pi\chi(A) &= \sum \kappa(v) + \sum \kappa(f) \\ 0 &= \sum \kappa(v) + \sum \kappa(f) \\ 0 &\leq \pi\#_S + \pi\#_0 + \pi\#_1 + \tfrac{2\pi}{3}\#_2 + \tfrac{\pi}{3}\#_3 + 0\#_4 + 0\#_C \end{aligned}$$

(where $\#_C$ stands for number of connector - see figure)

(figure: connector)

If A is a $C'(\frac{1}{8})$ annular disc diagram, then either:

• A contains a spur, 0-shell, 1-shell, 2-shell, or 3-shell (in which case A can be simplified by shortening R_1 or R_2), or

• A has a very simple structure:

(figure: list of simple possibilities)

This solves the conjugacy problem.