Theorem 1. *(Greendlinger's Lemma) Let $D \to X$ be a $C(6)$ [$C(4) - T(4)$] disc diagram. Then either*

1) D is trivial
2) D is a single 2-cell
3) [$C(6)$] D contain a "count" of 6 i-shells/spurs, where
spurs, 0-shells, 1-shells count as 3,
2-shells count as 2,
3-shells count as 1.
3) [$C(4) - T(4)$] D contain a "count" of 4 i-shells/spurs, where
spurs, 0-shells, 1-shells count as 2,
2-shells count as 1.

(figure: examples)

PROOF of [$C(6)$]. Assume D is not a single 2-cell or 0-cell. Ignore all valence 2 0-cells. Assign angles as follows:

(figure) totally internal angles $\frac{2\pi}{3}$,
(figure) half internal angles $\frac{\pi}{2}$,
(figure) totally external angles 0.

Now D is an angled 2-complex.

1) Each internal 0-cell has curvature ≤ 0.

2) Each external 0-cell has curvature ≤ 0 except endpoint of a spur has curvature $= \pi$.

Indeed, if $v \in \partial D$ but valence$(v) \neq 1$, then the link of v is a forest with at least 3 vertices. Therefore
either link(v) has at least 2 components, so $\chi(\text{link}(v)) \geq 2$,
or link(v) is connected and $\sum \angle \geq \pi$ (figures of 3 possibilities).

So 2-cells must make positive curvature. Each internal 2-cell has non-positive curvature (sum of angles $\geq \frac{2\pi}{3} \cdot 6$). Each 2-cell with ≥ 2 boundary arcs has curvature ≤ 0, since has ≥ 4 angles of value $\frac{\pi}{2}$.

EXERCISE. Check this.

(figures: possible cases)

Remaining 2-cells have exactly 1 boundary edge. Curvature of a 2-cell is $\kappa(f) = \sum \angle - (|f| - 2)\pi$. Sum of angles of an i-shell ($i \geq 1$) is $\sum \angle = \frac{\pi}{2} + \frac{\pi}{2} + (i-1)\frac{2\pi}{3}$. Then $\kappa(f) = \pi - \frac{\pi}{3}(i-1)$

$\kappa(1\text{−shell}) = \pi$
$\kappa(2\text{−shell}) = \frac{2\pi}{3}$
$\kappa(3\text{−shell}) = \frac{\pi}{3}$
$\kappa(4\text{−shell}) = 0$

(By the way: you may have as many 4-shells as you like.)

$$2\pi\chi(D) = \sum \kappa(v) + \sum \kappa(f)$$

$$\begin{array}{ccccccccccccc} 2\pi & \leq & 0 & + & \pi \cdot \#_S & + & \pi \cdot \#_0 & + & \pi \cdot \#_1 & + & \frac{2\pi}{3} \cdot \#_2 & + & \frac{\pi}{3} \cdot \#_3 \\ 6 & \leq & 0 & + & 3 \cdot \#_S & + & 3 \cdot \#_0 & + & 3 \cdot \#_1 & + & 2 \cdot \#_2 & + & 1 \cdot \#_3 \end{array}$$

PROOF SKETCH for $C(4) - T(4)$ case. Similar argument except use $\frac{\pi}{2}$ angles on internal and half-internal corners, 0 angles on totally external corners (ends up $\kappa(f) = (3 - i)\frac{\pi}{2}$). $\square$

Why no lemma for $C(3)-T(6)$? (figure: hexagon) Zillion 2-shells, no 1-shells, no 0-shells. Although, there are positive features. (figure: rhombus).

EXERCISE. Give example of infinitely mane $(6,3)$ complexes.

Rips's Short Exact Sequence. A construction producing interesting subgroups of $C'(\frac{1}{6})$ groups.

Let Q be a finitely presented group $\langle \bar{g}_1,\dots,\bar{g}_s|\overline{R}_1,\dots,\overline{R}_t\rangle$. There exist a short exact sequence

$$1\to N\to G\to Q\to 1$$

s.t. G is a finitely presented $C'(\frac{1}{6})$ groups, N is finitely generated.

$$1\to\langle x,y\rangle\to\langle x,y,g_1,\dots,g_s|R_1,\dots,R_t\rangle\to\langle \bar{g}_1,\dots,\bar{g}_s|\overline{R}_1,\dots,\overline{R}_t\rangle\to 1$$

Rips's construction can be used to produce $C'(\frac{1}{6})$ group G with interesting subgroups from a group Q with interesting properties.

EXAMPLE. If Q has undecidable word problem then G has undecidable membership problem in N. Induced from a word $\overline{w}$ in generators of Q, $(\overline{w}=1)\Leftrightarrow(w\in N)$.

EXAMPLE. If has a finitely generated subgroup $\overline{H}$ that is finitely generated but not finitely presented then its preimage H in G is finitely generated but not finitely presented.

Indeed, if $\overline{H}=\langle\overline{w}_1,\dots,\overline{w}_n\rangle$ then $H=\langle w_1,\dots,w_n,x,y\rangle$. If H is finitely presented then $H=\langle h_1,\dots,h_k|U_1,\dots,U_m\rangle$, so $\overline{H}=\langle h_1,\dots,h_k|U_1,\dots,U_m,X,Y\rangle$.