EXAMPLE. $\langle a, b | abab^2ab^3 \cdots ab^{100}, baba^2ba^3 \cdots ba^{100} \rangle$. Symbols ab^na, ba^nb occur uniquely for $n > 1$. Thus largest piece is smaller than $b^{99}ab^{100}, a^{99}ba^{100}$, so has length < 200. But each relator has length 5150. Therefore X satisfies $C'(\frac{200}{5150}) \Rightarrow C'(\frac{1}{25}) \Rightarrow C(26) \Rightarrow C(7)$

EXAMPLE. $\langle a, b | W^n \rangle$ satisfies $C'(\frac{1}{n}), C(n+1)$, as any piece has length $< |W|$. (A longer piece would imply $W = V^m$ for some $m > 1$.)

Lemma 1. *A positive presentation satisfies $T(4)$. (All relators are positive words in generators, e.g. $\langle a, b, c | abac, bbaabccb \rangle$.)*

More generally, if X is a 2-complex and X^1 is a directed graph, and all attaching maps are directed cycles, then X satisfies $T(4)$ condition.

PROOF. There can't be any valence 3 vertices.

$\langle a, b | a^nbab^2ab^3 \cdots ab^{100} \rangle, n \to \infty$ satisfies $C(p)$ with p constant, $C'(\alpha)$ with $\alpha \to 1$. So in a sense $C(p)$ condition is far more general than $C'(\alpha)$.

EXAMPLE. $\langle a, b | R_1, \ldots, R_m \rangle$. Fix m. Suppose each R_i has length $< n$. Then presentation satisfies $C'(\alpha), \alpha \to 0$ as $n \to \infty$.

Thus $C'(\frac{1}{6})$ presentations are quite common for presentations with few long relators. $T(4)$ is rare.

Lemma 2. *$T(5) \Rightarrow$ all pieces have length 1.*

PROOF. Suppose R_1, R_2 share piece of length ≥ 2. Then there is a degree 2 branch cover which fails $T(5)$ condition.

(figure: degree 2 branch cover)

So, for example, $T(5), C(10) \Rightarrow C'(\frac{1}{10})$.

$T(5)$ condition is rather rare.

Lemma 3. *For $q \geq 5$ X is $T(q)$ iff each $\mathrm{link}(v)$ of each 0-cell has girth $\geq q$. (More generally, we can require "all pieces have length 1" instead of "X is $T(q \geq 5)$"*

Theorem 1. *If X is compact and finite 2-complex and either of $\{C(3)-T(7), C(4)-T(5), C(5)-T(4), C(7)-T(3)\}$ then $\pi_1 X$ has a linear isoperimetric function, thus word-hyperbolic.*

PROOF. Any reduced disc diagram $D \to X$ satisfies a (p, q) condition after ignoring valence 2 vertices in its interior, and subdividing the boundary a bounded amount. (As X satisfies negative weight test: $\frac{1}{3} + \frac{1}{7} > \frac{1}{2}, \frac{1}{4} + \frac{1}{5} < \frac{1}{2}$). Thus $\mathrm{Area}\,(D) \leq k|\partial_p D|$ for some constant $k = k(X)$.

(figure: subdividing)

Amount of new vertices is bounded by $|\partial_p D| \cdot p \leq 6|\partial_p D|$. Subdividing didn't change area, so

$$\mathrm{Area}\,(D) = \mathrm{Area}\,(D') \leq k|\partial_p D'| \leq 6|\partial_p D|.$$

Theorem 2. *X is either of $\{C(3)-T(6), C(4)-T(4), C(6)-T(3)\}$ and compact then X has at most quadric isoperimetric function.*

PROOF. Analogous.

Dehn Presentation.

Definition 1. *$G = \langle a_1, \ldots, a_m | R_1, \ldots, R_n \rangle$ is a Dehn presentation if for each word P with $P =_G 1_G$ there either*

1) P has a backtrack, or
2) P has a subpath Q such that

- *$QS =$ some cyclic permutation of $R_i^{\pm 1}$*
- *$|Q| \geq |S|$ i.e. $|Q| \geq \frac{1}{2}|R_i|$.*

(figures: cases 1 and 2)

Theorem 3. *If $\langle a_1, \ldots, a_m | R_1, \ldots, R_n \rangle$ is a Denh presentation then its group G has linear isoperimetric function.*

PROOF. Proved by induction that Area $(P) \leq |P|$. Let P ba a word $P =_g 1_G$.
Base case: $0 = \text{Area}\,(1)\,, 0 = |1|$.
By hypothesis either
1) $P = Aee^{-1}B$, where $e = a_i^{\pm 1}$, or
2) $P = AQB$, where $Q \subset \partial R_i, |Q| > |R_i|$.

Then $P' = AB$ or $P' = AS^{-1}B$ respectively also represents 1_G. $|P'| < |P|$, therefore Area $(P') \leq |P'|$ by induction. Let D' be a reduced disc diagram for P'. Then

$$\text{Area}\,(P) \leq \text{Area}\,(P') + 1 \leq |P'| + 1 \leq |P|,$$

because to obtain a diagram D for P from D', we add a spur in case 1) and add a copy of R_i along S in case 2).
(figures: adding spur/cell)

Definition 2. *i-shell is a 2-cell R in disc diagram D whose boundary such that $|\partial_p R = QS$ where Q is a subpath of $\partial_p D$ and S is an internal path (i.e. no external 1-cells in S) which is concatenation of exactly i pieces.*
A spur is a 1-cell ending at valence 1 cell.

(figures: i-shells, spur)

Theorem 4. *(Greendlinger's Lemma) Any nontrivial $C(6) - T(3)$ disc diagram has at least two i-shells or spurs ($i \leq 3$).*
Any nontrivial $C(4) - T(4)$ disc diagram has at least two i-shells or spurs ($i \leq 2$).
("Nontrivial" means it's not a single point or loop.)

Corollary 1. *If X is a $C'(\frac{1}{4}) - T(4)$ or $C'(\frac{1}{6}) - T(3)$ complex, then it is a Dehn complex (presentation).*

PROOF of Corollary. Let P be a nullhomotopic path in X. Let D be a minimal area disc diagram $D \to X$. By Greendlinger's Lemma, if D is nontrivial, it has 2 i-shells or spurs, so at least one i-shell or spur away from a base point.
1) Spur has a backtrack.
2) i-shell with interior subpath S and exterior subpath Q. (figure: i-shell)
$C'(\frac{1}{6}) \Rightarrow i \leq 3$, then $|S| \leq |P_1| + |P_2| + |P_3| < \frac{1}{6}|R| + \frac{1}{6}|R| + \frac{1}{6}|R| = \frac{1}{2}|R|$, thus $|S| < |Q|$,
$C'(\frac{1}{4}) \Rightarrow i \leq 2$, then $|S| \leq |P_1| + |P_2| < \frac{1}{4}|R| + \frac{1}{4}|R| = \frac{1}{2}|R|$, thus $|S| < |Q|$.