Small Cancellation Theory.

Definition 1. *Let $D \to X$ be a disk diagram in some 2-complex. A cancelable pair of 2-cells in D is a pair of 2-cells R_1, R_2 meeting along a 1-cell e in D such that ∂R_1, ∂R_2 starting at e project to the same path in X.*

For instance, if R_1, R_2 fold together along the way to X, the R_1, R_2 are the cancelable pair.

(figure: a near-immersion with a cancelable pair)

When a presentation has duplicate relators (e.g. $\langle a, b | abb, abb, babaa\rangle$) it is natural to have near-immersions with cancelable pairs. With $X = \langle a, b | (ab)^3\rangle$, $\widetilde{X}$ has many duplicate 2-cells.

(figure: tree of hexagons)

EXERCISE. Show X has an index 6 subgroup that is free (degree 6 covering map).

Definition 2. *A disc diagram $D \to X$ is reduced if it has no cancelable pairs.*

NOTE. If X has no duplicate 2-cells, no 2-cells with periodic attaching maps, then disc diagrams in X are reduced iff $D \to X$ is a near-immersion.

EXERCISE. If a disc diagram satisfies negative weight test, then it has no cancelable pairs.

NOTE. Any minimal area disc diagram with boundary path p is reduced, since we can remove cancelable pairs to decrease area. Converse does not hold. Relationship in 2 dimensions is not understood, counterexamples are not aspherical. (figure: cubes)

Small Cancelation Conditions. The maximal internal arcs in reduced disc diagrams are called pieces.

Definition 3. *X satisfies $C(p)$ condition if for any reduced disc diagram $D \to X$ no boundary path of a 2-cell R in D is concatenation of $< p$ pieces.*

In other words, reduced daisy with less than p petals doesn't exist.

(figure: daisy flower)

If X is $C(p)$ then in any reduced disc diagram $D \to X$ when we ignore valence 2 vertices in D, each 2-cell of D without boundary (lying in ∂D) 1-cells looks like $\geq p$-gon.

(figure: rectangular figure) X might be $C(4)$, not $C(5)$.

Definition 4. *X satisfies $T(q)$ condition if each reduced disc diagram $D \to X$ has the property thattach internal 0-cell $v \in D$ satisfies either $\deg v = 2$ or $\deg v \geq q$.*

(figure: pizza pie) Equivalently, any reduced pizza pie $\to X$ has 2 slices, or $\geq q$ slices.

These definitions usually presume that attaching maps of 2-cells are immersions.

EXAMPLE. $\langle a, b | abb^{-1}\rangle$

(figure: one-arrowed clock) $T(2)$ — attaching maps are immersions.

NOTE. $T(q)$ implies immersed attaching maps for $q \geq 2$.

We refer to a presentation as $C(p) - T(q)$ if its standard 2-complex satisfies $C(p) - T(q)$.

NOTE. A piece in a presentation is a word p that appears in "more than one way" among relators.

EXAMPLE. $\langle a, b | abbaaabbaaab \rangle$ — aa, $abbaaab$.

$\langle a | aaaaaaa \rangle$ is $C(\infty) - T(\infty)$. For instance, a diagram corresponding to piece aaa is not reduced. (figure: a diagram)

$\langle a, b, c | (abc)^{100} \rangle$ is $C(\infty) - T(\infty)$.

EXERCISE. Every group has a $C(5)$ presentation (also has $C(4) - T(3)$ and $C(3) - T(4)$ presentations). It's about subdivision.

Definition 5. *X satisfies $C'(\alpha)$ condition if for each piece p occurring in $\partial_p R$ we have $|p| < \alpha |\partial_p R|$. (This is a metric small-cancelation condition.)*

NOTE. $C'(\frac{1}{n}) \Rightarrow C(n+1)$.

EXAMPLE. $\langle a, b | abab^2ab^3 \cdots ab^{100}, baba^2ba^3 \cdots ba^{100} \rangle$. A piece is not longer than $b^{98}ab^{99}$, so it's $C'(\sim \frac{4}{99}) \Rightarrow C(6)$.