# Lecture 11

Last time: Let $D$ be a disc diagram, $(4,4)$-comlex. Girth $\geq 4$, each 2-cell is square. Then $\text{Area}(D) \leq |\partial_P D|^2$

Observe. Let $X$ be $(4,4)$-complex, its first cubical subdivision is a $(4,4)$-complex whose 2-cells are squares.

(figure: cubical subdivision) "old" links did not change. New ones have girth at least 4. So cubical subdivision preserves "(4,4)".

$X' = \text{subdivide}\, X$. Isoperimetric function are $\sim$ the same. Therefore we have the following theorem.

**Theorem 1.** *Let $X$ be finite $(4,4)$-complex. Then its isoperimetric function $f(n)$ is $\leq kn^2$.*

**Theorem 2.** *Let $X$ be finite $(3,6)$-complex or $(3,6)$-complex. Then its isoperimetric function $f(n)$ is $\leq kn^2$.*

**Theorem 3.** *Let $D$ be a $(3,6)$ disc diagram whose 2-cells are triangles. Then* $\text{Area}(D) \leq |\partial_P D|^2$.

Proof. As usual, consider non-singular disk diagram $D$. We consider the dual curves in $D$ modeled on

(figure: types of odd/even dual curves)

**Lemma 1.** *No dual curve is a circle. (figure)*

**Lemma 2.** *No dual curve self-intersects. (figure)*

**Lemma 3.** *No two dual curves intersect in a bigon. (figure)*

Proof. Barycentrycally subdivide $D$ to form $D'$, so dual curves occupy 1-skeleth of $D'$. Apply combinatorial Gauss-Bonet theorem to $E$, conclude $E$ has $\geq 3$ corners.

(figure: $E$ inside $D$)

$$\begin{array}{rcccccc} 2\pi\chi(E) & = & \sum_f \kappa(f) & + & \sum_{v\in\partial E} \kappa(v) & + & \sum_{v\in \text{Int} E} \kappa(v) \\ 2\pi & \leq & 0 & + & \frac{5\pi}{6}\begin{pmatrix}\text{number}\\ \text{of}\\ \text{corners}\end{pmatrix} & + & 0 \end{array}$$

Note. $D'$ satisfies non-positive weight test in an obvious way.

(figure: contributions of acute corners)

Only acute corners give positive contribution. Number of acute corners $\geq \frac{12}{5}$. Thus lemmas 1, 2, 3.

Back to proving theorem. Bound number of 2-cells by number$\cdot$(dual curves)$^2$.

(figure: dual curves corresponding to triangle)

$$\{\text{triangles}\} \xrightarrow{h} \{\text{dual curves}\}^2$$

$h$ is injective by lemma 2.

$\{\text{dual curves}\} \to \{\text{cells in} \partial_P D\}$

We are done once $\{\text{dual curves}\} \leq k|\partial_P D|$.

(figure: how dual curves behave when hit the boundary)

When curve hits the boundary vertex, the 6 middle ones stop, the rest are continued symmetrically. So $\{\text{dual curves}\} \leq 14|\partial_P D|$.

We proved that for $D$ a triangular $(3,6)$ disk diagram, $\text{Area}(D) \leq |\partial_P D|^2$.

**Theorem 4.** *Let $X$ be $(3,6)$-complex or $(6,3)$-complex. Then the isoperimetric function is bounded by a quadratic function $f(n) \leq kn^2$.*

Proof. First triangle subdivision of $(6,3)$-complex is a $(3,6)$-complex. Moreover, any $(3,6)$-complex can be subdivided into a triangular $(3,6)$-complex, meaning all 2-cells are triangles. (Just divide each $n$-gon into $(n-2)$ triangles without new 0-cells.) But a triangular $(3,6)$-complex has $f(n) \leq n^2$, because this holds for any triangular $(3,6)$-diagram. That completes the proof.

□

**Construction of "thick" $(3,6)$-complexes.** ("Thick" means each 1-cell has at least three 2-cells attached locally.)

Branched covering spaces. (figures)

$X = \langle a | aaa^{-1} \rangle$, $Y = X \setminus D_\epsilon(p)$.

$\pi_1 Y$ is free, so $Y$ has many finite covering spaces. Choose a finite cover $\hat{Y} \to Y$. Then add in the points $p$ to form a branch cover $\hat{X} \to X$. If you choose $\hat{Y}$ appropriately (large enough), then $X$ is $(3,6)$-complex, $\hat{X}$ is always thick.

(figures: link of the 0-cell, $Z$ in $Y$)

$Y = X \setminus p$

Deformation retraction $Y \to Z$ yields immersion link$(p) \to Z$.

Covering spaces of $Z \longleftrightarrow$ covering spaces of $Y \longleftrightarrow$ branch covering spaces of $X$

Deformation retraction $Y \to Z$

$\updownarrow$

Deformation retraction $\hat{Y} \to \hat{Z}$

induces immersion link$(\hat{p}) \to \hat{Z}$.

Thus to make $(3,6)$-complex $\hat{X}$ choose $\hat{Z}$ to have girth $\geq 12$ (any graph has a finite cover with a given girth).