# Lecture 10

*Reviewing end of Lecture 9.*

**Lemma 1.** *Let $\Gamma$ be a finite angled graph, whose angle girth is $\geq 2\pi$, i.e. $|\sigma|_\angle \geq 2\pi$ for each immersed closed path $\sigma$. Then there exists a constant $c_2 \leq 0$ such that $|\gamma|_\angle \geq \pi - c_2$ for arbitrary immersed path $\gamma \to \Gamma$.*

Proof. (figure) Divide $\gamma$ into closed paths and two "tails", which give some bounded negative contribution.

Exercise. Write the proof.

**Lemma 2.** *If $X$ satisfies negative weight test and $X$ is compact, then $\exists K = K(X)$ such that $\text{Area}\,(D) \leq K|\partial_P D|$ for any near-immersed $D \to X$.*

Proof. Without loss of generality assume $D$ is non-singular. Pull back angles to $D$. Apply combinatorial Gauss-Bonet theorem.

$$2\pi = 2\pi\chi(D) = \sum_f \kappa(f) + \sum_{v \in \partial D} \kappa(v) + \sum_{v \in \text{Int}\,D} \kappa(v)$$

(figure: $D$)

Let $c_1$,$c_2$ be such that $\kappa(f) \leq -c_1 < 0$, $c_2$ as in lemma 1 applied to any link$(x), x \in X^0$. $|\gamma|_\angle \geq \pi - c_2$, so $\kappa(v) = \pi - |\gamma|_\angle \leq c_2$.

Now we have

$$\begin{aligned} 2\pi &\leq c_1 \text{Area}\,(D) + c_2|\partial_P D| + 0 \\ c_1 \text{Area}\,(D) &\leq c_2|\partial_P D| - 2\pi \\ \text{Area}\,(D) &\leq \frac{c_2}{c_1}|\partial_P D| \end{aligned}$$

□

Consider $\langle a_1, a_2, \ldots, a_g | a_1^2 a_2^2 \cdots a_g^2 \rangle$, whose standard complex is genus $2g$ non-orientable surface. It is $(2g, 2g)$-complex, and corresponding disk diagram is a $2g$-gon.

$g = 2$ $(4,4)$-complex — nonpositive weight test,

$g \geq 3$ $(2g, 2g)$-complex — negative weight test.

Exercise. Let $0 < g < h$ and $\Sigma_g$ denote genus $g$ surface. There is no injection $\pi_1 \Sigma_g \hookrightarrow \pi_1 \Sigma_h$ (both surfaces orientable or both non-orientable).

**Dehn Complex of a Link.**

Example. (Max Dehn) Naturally occuring $(4,4)$-complexes.

(figure: knot)

Knot is embedding $S^1 \hookrightarrow \mathbb{R}^3$, or $S^1 \hookrightarrow S^3$. (Smooth or piecewise linear.)

Link is embedding $\bigsqcup S^1 \hookrightarrow \mathbb{R}^3$. $\pi_1(S^3 \setminus L)$ is a nice presentation of a link.

(figure: knot over a plane) Imagine link $L$ covering above or below a "projection plane" $\mathbb{R}^2 \subset \mathbb{R}^3$.

On the plane, triple crossings are not allowed (see figure)

(figure: allowed crossing, forbidden crossing)

Since valency of each vertex is 4, projection has a checkerboard coloring.

Dehn complex $D$ is determined by the following:

two 0-cells $\alpha, \beta$ above and below $\mathbb{R}^2$;

1-cells correspond to regions in projection;

2-cells correspond to crossing points in projection. They come in two flavors, depending on coloring of crossing point neighborhood (see figure). 2-cells are attached to four 1-cells each as shown (see figure).

(figure: two types of crossing)
(figure: attaching a 2-cell)
(figure: Dehn complex)

**Theorem 1.** *$D \hookrightarrow (\mathbb{R}^3 \setminus L)$ induces a $\pi_1$-isomorphism $\pi_1 D \to \pi_1(\mathbb{R}^3 \setminus L)$.*

**Definition 1.** *Link projection is alternating, if each circle alternatively travels above and below at crossing.*

(figure: alternating link projection)

**Definition 2.** *Link projection is prime, if every circle $C$ in $\mathbb{R}^2$, which intersects the projection in two regular points, has a simple arc on one side.*

(figure: prime link projection)
A projection is not prime iff it is a "connected sum" of two projections.

**Theorem 2.** *If projection is prime and alternating, then Dehn complex $D$ is $(4,4)$-complex.*

PROOF. Each 2-cell has $\geq 2$ sides. We need to check that $\mathrm{link}(\alpha), \mathrm{link}(\beta)$ have girth $\geq 4$. Examine $\mathrm{link}(\alpha)$ ($\mathrm{link}(\beta)$ similar).
(figure)
In fact, $\mathrm{link}(\alpha)$ inbeds in the projection picture. Checkerboard coloring gives bipartile structure to $\mathrm{link}(\alpha)$. Therefore, there are no sycles of length $1, 3$. Then girth $\geq 4$ if no bigons. Two types of bigons:
(figure) — not alternating
(figure) — not prime
EXERCISE. Think through logical details.

□

4-valent graph → make alternating projection → Dehn $(4,4)$-complex.

**Quadratic isoperimetric function for $(4,4)$-complex.** Let $D$ be a disc diagram, whose 2-cells are squares, 0-cells have links of girth $\geq 4$. $D$ has a system of "dual curves", each curve passing through opposite sides of squares (see figure).
(figure: disc diagram with dual curves)

**Lemma 3.** *No dual curve is a circle. (figure)*

**Lemma 4.** *No dual curve self-intersects. (figure)*

**Lemma 5.** *No two dual curves intersect in a bigon. (figure)*

**Lemma 6.** *No two three curves intersect in a triangle. (figure)*

PROOF. Subdivide $D$ along dual curves. Consider a subdiagram $E$ bounded by dual curves. Note: $E$ is a $(4,4)$-complex. Apply combinatorial Gauss-Bonet theorem to $E$:

$$\begin{aligned} 2\pi\chi(E) &= \sum_{v \in \partial E} \kappa(v) + \sum_{v \in \mathrm{Int} E} \kappa(v) + \sum_f \kappa(f) \\ 2\pi \leq &= \sum_{v \in \partial E} \kappa(v) + (\leq 0) + 0 \end{aligned}$$

Conclusion: $E$ has at least four $\frac{\pi}{2}$ corners.

Finally we show that Area $(D) \leq (|\partial_P D|)^2$ (without loss of generality $D$ is non-singular). Each 2-cell in $D$ gives distinct dual curves (by lemma 4). By lemma 3, these curves hit the boundary. By lemma 5, the map {2-cells} $\rightarrow$ {distinct pairs of dual curves} is injective.

(figure: square and corresponding dual curves)

This gives inequality:

$$\text{Area}\,(D) \leq \binom{\frac{\partial_p D}{2}}{2} \leq |\partial_P D|^2$$

□

Exercise. Find an interesting (thick) $(3, 6)$-complex.