Lecture 9

Definition 1. *An immersion $Y \to X$ is a local injection. A near-immersion $Y \to X$ is a map such that its restriction to $Y \setminus Y^0$ is an immersion.*

Note. A combinatorial map $\varphi : Y \to X$ is an immersion iff $\varphi : \text{link}(v) \to \text{link}(\varphi(v))$ is an embedding.

It is a near-immersion iff $\varphi : \text{link}(v) \to \text{link}(\varphi(v))$ is an immersion. (Also, it is the same being embedding of links of 1-cells.)

(figure: six slices pie mapping onto three slices pie)

Given a map of a 1-cell, we can continue it on the corresponding 2-cell by coning

$$Y \rightsquigarrow {}^{Y \times [0,1]} \Big/ {}_{Y \times \{1\} = c}$$

(figures: cones)

Lemma 1. *Suppose $Y \to X$ is a near-immersion, suppose also X is an angles 2-complex which satisfies the [negative] weight test. Then Y satisfies the same condition.*

Proof. Pull back the angles on corners of X to corners of Y. Now the map $\varphi : Y \to X$ is an angle preserving map between angled 2-complexes. $\kappa(f) \leq 0$ for each 2-cell of Y.

Let σ be immersed closed path in in $\text{link}(y)$, $y \in Y^0$. Then $\varphi \circ \sigma$ immersed closed path in in $\text{link}(\varphi(y))$, $\varphi(y) \in X^0$, and $|\sigma|_{\angle(Y)} = |\varphi \circ \sigma|_{\angle(X)} \geq 2\pi$.

Corollary 1. *Let X satisfy [negative] weight test. If $S \to X$ is a near-immersion of a close surface then $\chi(S) \leq 0$ [$\chi(S) < 0$],*
so there are no near-immersion $S^2 \to X$ [$T^2 \to X$].

or, in other words,

Corollary 2. *X is aspherical, i.e. $\pi_i X = 1$ for $i \geq 2$, i.e. universal cover of X is contractible [$\mathbb{Z}^2 \not\hookrightarrow \pi_1 X$, i.e. non-hyperbolic].*

Definition 2. *X is diagrammatically reducible (DR) if there is no near-immersion $S^2 \to X$.*

Conjecture. (Whitehead) Let X, a 2-complex, be aspherical. Let A be a subcomplex. Then A is aspherical.

Theorem 1. *If X is DR then X is aspherical.*

Idea of proof. Let $S^2 \xrightarrow{f} X$. Simplitial approximation theorem: allows to choose a subdivision of S^2, X and a new cellular map $g : S^2 \to X$ such that $f \simeq g$ (homotopic). You can snush/ignore cells that get squashed. Left with $h : S^2 \to X$ combinatorial, $h \simeq g \simeq f$. But h is not yet near-immersion.

Let Y be a surface. Suppose $Y \to X$ is a near-immersion.

(figure: removing a "cancellable pair")

(figure: dealing with deviant cases)

Idea. Suppose X satisfies the negative weight test. Suppose $\mathbb{Z}^2 \hookrightarrow \pi_1 X$. There exists a map $T^2 \to X$, $\pi_1 T^2 \to \pi_1 X$, $a \mapsto p_a, b \mapsto p_b$, closed paths in X, $\mathbb{Z}^2 = \langle p_a, p_b \rangle$.

(figure: a cell corresponding to a, b in disc diagram) It can't be singular, as p_a, p_b generate $\mathbb{Z}^2$.

Futzing around obtain a near-immersion $T^2 \to X$. Pull back angles. Apply combinatorial Gauss-Bonet theorem: $0 = 2\pi\chi(T^2) = \sum \kappa(v) + \sum \kappa(f) < 0$.

Isoperimetric Function.

Theorem 2. *Let X be a 2-complex. Let $P \to X$ be a nullhomotopic path. Then there exists a disc diagram $D \to X$ for P such that the map $D \to X$ is a near-immersion.*

PROOF. There exists some disc diagram $D_i \to X$ for P. If $D_i \to X$ is not a near-immersion, then D_i has a cancelable pair. Removing this cancelable pair yeilds a lower area disc diagram $D_{i+1} \to X$ with boundary path P. Eventually reach a near-immersion $D \to X$.

(However, disc diagram has not necessarily the minimal square:

figure: a cube — area 1 and area 5 diagrams.)

Theorem 3. *Let X be a compact angled 2-complex satisfying negative weight test. Then X has a linear isoperimetric function.*

PROOF. There exists a constant $M = M(X)$ such that for any near-immersion $D \to X$, $\text{Area}\,(D) \leq M|\partial_P D|$.

Let c_1 be such that $\kappa(f) \leq -c_1 < 0$ for all f in X.

Let c_2 be such that $|\sigma|_\angle > 2\pi + c_2$ for any σ.

Let $D \to X$ be a nonsingular dias disc diagram (singularity just increase length of boundary without increasing area). Pull back angles to D.

$$\begin{array}{rcccccc} 2\pi\chi(D) & = & \sum_f \kappa(f) & + & \sum_{v\in\partial D} \kappa(v) & + & \sum_{v\in \text{Int} D} \kappa(v) \\ 2\pi & \leq & \text{Area}\,(D)\cdot(-c_1) & + & c_2|\partial D| & + & 0 \\ & & c_1\text{Area}\,(D) & \leq & c_2|\partial D| & & \\ & & \text{Area}\,(D) & \leq & \frac{c_2}{c_1}|\partial D|. & & \end{array}$$

Done.